PARAMETRIC INTERACTION OF ELECTROMAGNETIC OSCILLATIONS IN NONLINEAR DIELECTRIC RESONATORS: BASIC EQUATIONS

G. V. Belokopytov

Vestnik Moskovskogo Universiteta. Fizika,
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The method of slowly-varying standing waves is used to obtain the truncated equations for the three-frequency resonance parametric interaction in nonlinear dielectric resonators. The equations are suitable for inhomogeneous and anisotropic media. Pump reaction and nonlinear detuning of both electrical and thermal character are taken into account.

The use of paraelectric resonators as nonlinear elements in parametric microwave amplifiers is discussed in [1-7]. The distributed character of regeneration in devices of this type has as its consequence a rapid reduction in the pump electric field as compared with amplifiers using lumped nonlinear paraelectric elements. It is suggested that this should result in a radical reduction in the level of nonequilibrium processes which give rise to a substantial enhancement of noise temperature [8, 9]. A reduction in the pump intensity would also ensure favorable conditions for a nonlinear element. Active dielectric resonators compare favorably with traveling-wave devices [10-12] in that they are easier to match to external circuits, they are simple to fabricate, and they require much lower pump power. Moreover, the combination of nonlinear and resonance properties in a single element (dielectric resonators) opens up new possibilities for the miniaturization of parametric amplifiers.

The properties of nonlinear interactions of different oscillation modes in dielectric resonators were analyzed in [1-4]. Parametric interactions between oscillations and resonators with distributed nonlinearity have also been examined in the theory of parametric light generators [13, 14], parametric amplifiers using ferrite [15], and distributed semiconductor generators [16-18]. In this paper, we derive the truncated equations which provide a very complete description of the dynamics of parametric interactions in nonlinear dielectric resonators. In contrast to the work reported in the papers cited above, the calculation is performed for an inhomogeneous and anisotropic active medium. The pump reaction is also taken into account. The use of the method of slowly varying standing waves enables us to take into account both linear and nonlinear detuning.

Theoretical analysis of electromagnetic oscillations in a resonator containing a nonlinear dielectric involves the solution of the macroscopic Maxwell equations

$$\mathrm{rot}\,\mathbf{H} = \frac{4\pi}{c} j + \frac{1}{c}\frac{\partial D}{\partial t}, \qquad (1)$$

$$\mathrm{rot}\,E = -\frac{1}{c}\frac{\partial B}{\partial t}. \qquad (2)$$

We shall take the constitutive relations between the electric fields, the inductions, and the currents in the form

$$E_i = \varkappa_{ik} D_k + \lambda_{ikl} D_k D_l + \nu_{iklm} D_k D_l D_m, \qquad (3)$$

$$j_i = j_i' + \rho_{ik} D_k + \theta_{ikl} D_k D_l. \qquad (4)$$

Here and henceforth, repeated indices represent summation unless indicated to the contrary. We shall suppose that medium is nonmagnetic, i.e., $B = \mu H$, where $\mu(r) = \mathrm{const}$. We have taken the constitutive relations in the form of (3) and (4) for the following reasons.

Firstly, the nonlinear dielectric properties of ferroelectric materials, including those in the paraelectric phase, are commonly characterized by expanding the thermodynamic potential in powers of polarization. For media with crystal lattice exhibiting cubic symmetry, we can write, by analogy with the Ginzburg-Devonshire expansion [19],

$$\Phi = \Phi_0 + \alpha D^2 + \beta_1 D^4 + \beta_2 (D_x^2 D_y^2 + D_y^2 D_z^2 + D_x^2 D_z^2) + \frac{1}{3}\gamma_1 D^6 +$$
$$+ \gamma_2 [D_x^4 (D_y^2 + D_z^2) + D_y^4 (D_x^2 + D_z^2) + D_z^4 (D_x^2 + D_y^2)] + \gamma_3 D_x^2 D_y^2 D_z^2.$$

where Φ_0, α, β_1, β_2, γ_1, γ_2, γ_3 are the expansion coefficients which, in general, are functions of temperature. Hence we can readily obtain the expression for the components $\varkappa_{ik}$, λ_{ikl} и ν_{iklm} in terms of the thermodynamic expansion coefficients. A constant bias field E_b is applied to the dielectric to produce an efficient nonlinear interaction. If in addition to the bias field, the dielectric experiences an alternating field $E_\sim$ ($|E_\sim| <<$ $<< |E_b|$), then at high frequencies the dielectric is a nonlinear medium characterized by the expansion given by (3). If both fields (E_b and $E_\sim$) are parallel to the x axis, we have

$$\varkappa_{11} = 2\alpha + 6\beta_1 D_b^2 + 10\gamma_1 D_b^4,$$
$$\lambda_{111} = 6\beta_1 D_b + 40\gamma_1 D_b^3,$$
$$\nu_{1111} = 2\beta_1 + 40\gamma_1 D_b^2.$$

The bias field is inhomogeneous in many structures that are convenient in practice. The coefficients in the expansion of the nonlinear permittivity are then functions of position.

Secondly, the nonlinear properties of resonators filled with paraelectric materials may be partly due to the dependence of the dielectric losses on the applied electric field. There is both theoretical and experimental evidence for this [20-22]. Finally, the constitutive equations written in the form given by (3) and (4) are convenient for the derivation of the truncated equations in the case of an inhomogeneous medium.

Since there is a great variety of resonator configurations (closed, open, with or without metal electrodes) we shall not specify the form of the boundary condition. The effect of extraneous forces will be described by a distributed current density j'. Since experiments show that there is no appreciable dispersion in the paraelectric phase up to frequencies of the order of 10^{12} Hz [23] in the case of dielectrics suitable for microwave applications (for example, strontium titanate), we shall neglect the frequency dependence of permittivity.

We must now solve the nonlinear problem defined by (1)-(4). We shall use the following notation: ϵ_{ikl} is the antisymmetric unit tensor (of Levi-Cevita) and ∂_k, which is the differentiation with respect to the k-th coordinate. From (1) and (2) we then have

$$\frac{c}{\mu} \in_{ikl} \partial_k \in_{lmn} \partial_m E_n + \frac{4\pi}{c}\frac{\partial}{\partial t} j_i + \frac{1}{c}\frac{\partial^2 D_i}{\partial t^2} = 0. \quad (5)$$

Substituting (5) in (3) and (4), we obtain

$$\frac{c}{\mu} \in_{ikl} \partial_k \in_{lmn} \partial_m (\varkappa_{nq} D_q + \lambda_{nqr} D_q D_r + \nu_{nqrs} D_q D_r D_s) +$$
$$+ \frac{4\pi}{c}\frac{\partial}{\partial t}(\rho_{iq} D_q + \theta_{iqr} D_q D_r + j_i') + \frac{1}{c}\frac{\partial^2 D_i}{\partial t^2} = 0. \quad (6)$$

We shall seek D_i in the form of a series in the eigenfunctions of the linear problem:

$$\frac{c}{\mu} \in_{ikl} \partial_k \in_{lmn} \partial_m \varkappa_{nq} D_{aq} = \frac{\omega_a^2}{c} D_{ai} \quad (7)$$

subject to the necessary boundary conditions. In these expressions, ω_a is the a-th eigenfrequency of the linear resonator (no summation over a in (7)). For normalized eigenfunctions in (7) we have the following orthogonality condition:

$$\int_V \varkappa_{ik} D_{ak} D_{bi} dV = \delta_{ab}.$$

We assume that there is no degeneracy with respect to type of oscillation in the resonator. The function $D_i(\mathbf{r}, t)$ in (6) will be taken in the form

$$D_i(\mathbf{r}, t) = p_a(t) D_{ai}(\mathbf{r}).$$

Multiplying (6) by $\varkappa_{iw} D_{fw}$ integrating over the volume of the resonator, and summing over a, we obtain the following equations, averaged over spatial variables:

$$\ddot{p}_f + \omega_f p_f = F_f(p|, \dot{p}|, p'(t)), \tag{8}$$

where

$$\begin{gathered}
F_f = -\dot{p}_a R_{fa} - (\dot{p}_a p_b + p_a \dot{p}_b)\Theta_{fab} - p_a p_b \Lambda_{fab} - p_a p_b p_c N_{fabc} - p_f'', \\
R_{fa} = 4\pi \int_V \varkappa_{iw} D_{fw} \rho_{iq} D_{aq} dV, \\
\Theta_{fab} = 4\pi \int_V \varkappa_{iw} D_{fw} \theta_{iqr} D_{aq} D_{br} dV, \\
\Lambda_{fab} = \frac{c^2}{\mu} \int_V \varkappa_{iw} D_{fw} \in_{ikl} \partial_k \in_{lmn} \partial_m \lambda_{nqr} D_{aq} D_{br} dV, \\
N_{fabc} = \frac{c^2}{\mu} \int_V \varkappa_{iw} D_{fw} \in_{ikl} \partial_k \in_{lmn} \partial_m \nu_{nqrs} D_{aq} D_{br} D_{cs} dV, \\
p_f'' = 4\pi \int_V \varkappa_{iw} D_{fw} \frac{\partial}{\partial t} j_i'(\mathbf{r}, t) dV.
\end{gathered} \tag{9}$$

Suppose there are three eigenfrequencies ω_1, ω_2, ω_3 in the resonator which are approximately synchronized so that $\omega_3 - \omega_1 = \omega_2(1 + \Delta_0)$, where Δ_0 is of the order of η ($\eta \ll 1$).

Suppose the system is subjected to external harmonic forces of frequency ω_H and ω_c:

$$\begin{gathered}
\omega_H = \omega_3(1 + \Delta_3); \ \omega_c = \omega_1(1 + \Delta_1), \\
p_1'' = 4\pi\omega_c J_c \cos(\omega_c t + \psi_c); \ p_3'' = 4\pi\omega_H J_H \cos\omega_H t.
\end{gathered}$$

We assume that the detunings Δ_1 and Δ_3 are small (Δ_1, $\Delta_3 \sim \eta$). Finally, let

$$\begin{gathered}
\omega_x = \omega_H - \omega_c, \\
\Delta_2 = (\omega_2\Delta_0 + \omega_3\Delta_3 - \omega_1\Delta_1)/\omega_2.
\end{gathered}$$

If the distributed system under consideration is nearly conservative, then the F_f on the right-hand side of (8) is of the order of η. We now apply the method of slowly varying amplitudes and seek the first-order solution of (8) in the form

$$p_f(t) = x_f(t)\cos(\tilde{\omega}_f t + \varphi_f(t)).$$

In this expression, x_f and φ_f are the "slow" amplitudes and phases and $\tilde{\omega}_f$ assumes the values ω_c, ω_x, ω_h. Using the usual transformations, we obtain the following truncated equations:

$$\begin{gathered}
2\omega_c\dot{x}_c + R_c\omega_c x_c + (\Lambda_c \sin\Psi - \omega_c\Theta_c\cos\Psi) x_H x_x = 4\pi\omega_c J_c \sin(\psi_c - \varphi_c), \\
2\omega_c\dot{\varphi}_c x_c - (\Lambda_c\cos\Psi + \omega_c\Theta_c\sin\Psi) x_H x_x -
\end{gathered} \tag{10a}$$

$$-\frac{x_c}{4}(3N_{cc}x_c^2+N_{cx}x_x^2+N_{cн}x_н^2+8\Delta_1\omega_c^2)=-4\pi\omega_c J_c\cos(\psi_c-\varphi_c), \qquad (10b)$$

$$2\omega_x\dot{x}_x+R_x\omega_x x_x+(\Lambda_x\sin\Psi-\omega_x\Theta_x\cos\Psi)x_н x_c=0, \qquad (10c)$$

$$2\omega_x\dot{\varphi}_x x_x-(\Lambda_x\cos\Psi+\omega_x\Theta_x\sin\Psi)x_н x_c-$$

$$-\frac{x_x}{4}(3N_{xx}x_x^2+N_{xc}x_c^2+N_{xн}x_н^2+8\Delta_2\omega_x^2)=0, \qquad (10d)$$

$$2\omega_н\dot{x}_н+R_н\omega_н x_н-(\Lambda_н\sin\Psi-\omega_н\Theta_н\cos\Psi)x_c x_x=-4\pi\omega_н J_н\sin\varphi_н,$$

$$2\omega_н\dot{\varphi}_н x_н-(\Lambda_н\cos\Psi+\omega_н\Theta_н\sin\Psi)x_c x_x- \qquad (10e)$$

$$-\frac{x_н}{4}(3N_{нн}x_н^2+N_{нc}x_c^2+N_{нx}x_x^2+8\Delta_3\omega_н^2)=-4\pi\omega_н J_н\cos\varphi_н. \qquad (10f)$$

where $\Psi=\varphi_н-\varphi_c-\varphi_x$; $\Lambda_c=\Lambda_{123}=\Lambda_{132}$; $\Lambda_x=\Lambda_{213}=\Lambda_{231}$; $\Lambda_н=\Lambda_{312}=\Lambda_{321}$; $\Theta_c=\Theta_{123}=\Theta_{132}$; $\Theta_x=\Theta_{213}=\Theta_{231}$; $\Theta_н=\Theta_{312}=\Theta_{321}$. In the expressions for R and N we have replaced subscripts 11, 22, 33 by c, x, h, respectively.

Apart from the usual assumptions with regard to the slow variation of the amplitudes, a very important condition for the validity of (10) is that only three types of waves are synchronous. In contrast to the nonlinear interaction of progressive waves, the general character of processes in the dielectric resonator is then found to depend not only on the ratio of the "length of formation of the discontinuity" and the length of coherent interaction [24], but also on the characteristic size of the resonator and the boundary conditions.

Equations (10) are formally analogous to the system of equations describing a parametric generator with three lumped circuits [25]. The quantity Λ then plays the role of the leading coefficients of the parametric coupling, the terms proportional to Θ are analogous to the decipitive amplitude limiting mechanism, and the terms proportional to N are analogous to the detuning mechanism in parametric semiconductor generators. However, as already noted, in contrast to systems with lumped parameters, the coefficients of the parametric interaction in nonlinear resonators are functions of the spatial distribution of the oscillation modes. This is determined by (9).

In nonlinear ferroelectric resonators there is one further specific mechanism that leads to the nonlinear limiting of oscillation amplitudes. Since the dielectric properties of a ferroelectric medium are temperature-dependent, high frequency heating due to losses leads to a change in the parameters of the medium. The most important is the change in the eigenfrequencies of the resonator from ω_f to the new values ω'_f:

$$\omega'_f=\omega_f(1-\delta_f).$$

Let us take the temperature dependence of susceptibility in the form

$$\varkappa_{ik}(\mathbf{r},T)=\varkappa_{ik}(\mathbf{r},T_0)+\left.\frac{d\varkappa_{ik}}{dT}\right|_{T=T_0}u,$$

where $u(r) = T(r) - T_0$ is a small change in the temperature and T_0 is the initial temperature. Using the variational expression for the eigenfrequencies of a resonator [26], we readily obtain

$$\delta_f=\frac{1}{2}\int_V u(r)\frac{d\varkappa_{ik}}{dT}D_{fk}D_{fi}dV$$

(here and henceforth we not sum over f). The temperature distribution in the resonator is given by the heat conduction equation

$$\frac{\partial u}{\partial t}=a^2\nabla^2u+F_1(\mathbf{r},t)$$

subject to the appropriate initial and linear boundary conditions. In these expressions, a is the thermal conductivity; $F_1(r, t)$ is the density of heat sources; $F_1(r, t) = $

$=\frac{1}{c\rho}\sum_f B_f(\mathbf{r})x_f^2(t)$, c is the specific heat; ρ is the density of the dielectric; $B_f=\frac{1}{2}\rho_{ik}D_{fk}\varkappa_{il}D_{fl}$.

It follows that transient processes in this system depend not only on the electrical but also on the thermal properties of the nonlinear resonator.

In the stationary case, the solution of the equation of thermal conduction is a superposition of solutions of the problems

$$\nabla^2 u_a + \frac{1}{k} B_a = 0$$

subject to the corresponding boundary condition, where k is the thermal conductivity. It is readily verified that $u(r) = u_a x_a^2$. Hence

$$\delta_f = n_{fa} x_a^2, \qquad (11)$$

where

$$n_{fa} = \frac{1}{2} \int_V u_a(\mathbf{r}) \frac{d\varkappa_{ik}}{dT} D_{fi} D_{fk} dV.$$

The change in the resonance frequencies must be taken into account in the truncated equations by replacing Δ_f with $\Delta_f + \delta_f$. It is clear from (10) and (11) that this is equivalent to replacing N_{fa} by N'_{fa}, where

$$N'_{fa} = N_{fa} + n_{fa} \left(8 - \frac{16}{3} \delta_{fa}\right) \omega_f^2 .$$

Thus, in the stationary case, the electrical and thermal detuning mechanisms are completely analogous. The differences between them appear only in the nonstationary case.

The system of equations given by (10) can be used to investigate the properties of a nonlinear dielectric resonator below and above the threshold for parametric self-excitation. Many results can then be obtained by using the analogy between the systems of truncated equations for distributed and lumped nonlinear elements. To investigate transient processes, we must in general take thermal inertia into account.

Strictly speaking, (10) does not take into account losses in electrodes or radiation losses, but for systems that are nearly conservative, all types of loss may be looked upon as independent and we can use the above system of equations with suitably modified numerical values of the coefficients R.

We note that a similar procedure can be used to obtain equations describing the parametric interactions under nonresonant pumping but the above resonant case is preferable because it enables us to achieve good matching between the resonator and the pump source [2].

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Department of Physics of Oscillations