THE DOUBLE INTEGRAL-TRANSFORM METHODS FOR FREDHOLM EQUATIONS OF THE FIRST KIND

V. P. Gostev and A. R. Frenkin

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We propose a method of solving (without investigating the questions of convergence and the possibility of readjusting the limits of integration) Fredholm (Volterra) integral equations of the first kind, which, if only formally, admit exact inversion. We describe a class of kernels to which one can apply the method of double integral transforms. Examples of exactly solved integral equations are given.

Fredholm integral equations of the first kind

$$\varphi(x) = \int_a^b K(x, y)\,\psi(y)\,dy, \tag{1}$$

where $\psi(x)$ is an unknown function and $\varphi(x)$ is assumed to be known, are encountered in various problems in mathematical and theoretical physics; for example, in the problem of diffraction on a half-plane [1] and in the quantum mechanical problem of transforming the wave function from one space to another [2]. In the general case Eq. (1) cannot be inverted exactly, and it is usually solved approximately [3]. Exact solutions of Eq. (1) have been obtained only for degenerate kernels $K(x, y) = \sum_{n=1}^{N} f_n(x)\,g_n(y)$ [4], for kernels which are Green functions of differential operators [1], for functions K(x, y) which are kernels of integral transforms (Fourier, Hankel, Laplace, etc.) [4], and for kernels which are functions of the difference K(x-y) [4]. By extremely refined methods exact solutions have been obtained for a special type of kernel (Abel and Shlemil'kh equations with a hypergeometric kernel, etc.) [4,5]. The example of kernels of the form K(x-y) in an infinite interval of integration ($a = -\infty$, $b = +\infty$) [1,4] suggests that Equation (1) can in many cases be reduced to an algebraic equation by applying suitable integral transformations. The method of solving equations (1) with kernels of the form K(x-y) by Fourier transformation of the kernel and the function $\varphi(x)$ is generalized below to the case of kernels and integral transforms of more general form.

We will assume that the transform of some integral transformation of the kernel K(x, y) with respect to x is factorable, i.e., contains as a multiplicative factor the kernel of the inversion of the other transformation (with respect to y). Of course, for such a generalization the kernal K(x, y) will not be a function of (x-y). We will examine a scheme for the inversion of Eq. (1) in this case. Let $s(\lambda, x)$, and $t(\lambda, x)$ be kernels of integral transforms with known inversion kernels $\bar{s}(\lambda, x)$, and $\bar{t}(\lambda, y)$ such that for "arbitrary" functions $\xi(x)$ and $\eta(x)$ the following reciprocal formulas hold:

$$\xi(x) = \int_p^q s(\lambda, x)\,\xi_s(\lambda)\,d\lambda, \tag{2}$$

$$\xi_s(\lambda) = \int_c^d \bar{s}(\lambda, x)\,\xi(x)\,dx, \tag{3}$$

$$\eta(x) = \int_k^l t(\lambda, x)\,\eta_t(\lambda)\,d\lambda,$$

$$\eta_t(\lambda) = \int_a^b \bar{t}(\lambda, x)\,\eta(x)\,dx. \tag{4}$$

If the s transform of the kernel of Eq. (1)

$$K_s(\lambda, y) = \int_c^d K(x, y)\,\bar{s}(\lambda, x)\,dx \tag{5}$$

with inverse

$$K(x, y) = \int_p^q s(\lambda, x)\,K_s(\lambda, y)\,d\lambda \tag{6}$$

is of the factorable form

$$K_s(\lambda, y) = \bar{t}(\lambda, y)\,f(\lambda), \tag{7}$$

then Eq. (1) can be written in the form

$$\varphi(x) = \int_a^b dy\,\psi(y) \int_p^q \bar{t}(\lambda, y)\,f(\lambda)\,s(\lambda, x)\,d\lambda. \tag{8}$$

After a change of the order of integration, the first part of Eq. (8) is the s expansion of $\varphi(x)$

$$\varphi(x) = \int_p^q \varphi_s(\lambda)\,s(\lambda, x)\,d\lambda,$$

where

$$\varphi_s(\lambda) = f(\lambda)\,\psi_t(\lambda),$$

$$\psi_t(\lambda) = \int_a^b \psi(x)\,\bar{t}(\lambda, x)\,dx \tag{9}$$

and the function $t(\lambda, y)$ in expansion (7) is continued naturally from the interval (p, q) to the interval (a, b) if necessary. The relation between the s transform of $\varphi(x)$ and the t transform of $\psi(x)$ is given by the simple formula (9), which reduces the integral equation (1) to an algebraic equation under condition (7). Inversion of (9) with the aid of formulas (3) and (4) gives a formal solution of Eq. (1) with the kernel (6), (7) in the form of two quadratures:

$$\psi(x) = \int_k^l d\lambda\,t(\lambda, x)\,f^{-1}(\lambda) \int_c^d \bar{s}(\lambda, y)\,\varphi(y)\,dy. \tag{10}$$

We will make several observations in regard to the above solution.

1. The upper limits of integration q, l for all practically applicable transformations are infinite ($q = l = \infty$), while the lower limits p, k can be finite (sine and cosine Fourier transforms, Laplace and Mellin transforms, etc.) or infinite: $p = k = -\infty$ (exponential Fourier transform). The limits of integration c, d, a, b in the inversions can be infinite (all upper limits) and can even lie in the complex plane (Mellin and Laplace transforms) [4,6].

2. The questions of the class of admissible functions $\varphi(x)$, the possibility of readjusting the limits of integration, convergence of the integrals, etc., should be examined for a specific choice of transformations $s(\lambda, x)$, $t(\lambda, x)$. The scheme of solution can be extended to generalized functions $\varphi(x)$, $\psi(x)$, and $K(x, y)$ [7].

3. With a known solution of Eq. (1) for kernel K(x, y) with transform (7), one can find a solution to the same equation with a kernel of the form

$$M(x, y) = \alpha(x)\beta(y)K(x,y). \tag{11}$$

This solution is obtained from solution (10) with the replacement $\varphi(y)\to\varphi(y)\alpha^{-1}(y)$, $\psi(x)\to\psi(x)\beta(x)$ and is of the form

$$\psi(x) = \beta^{-1}(x)\int_k^l f^{-1}(\lambda)t(\lambda, x)\,d\lambda\int_c^d \alpha^{-1}(y)\bar{s}(\lambda, y)\varphi(y)\,dy.$$

4. In the case when the transforms with respect to x and y are the same, i.e.,

$$s(\lambda, x) = t(\lambda, y),$$

the proposed method can be used for solving not only equations of the first kind, but also homogeneous equations of the second kind:

$$\psi(x) = \varphi(x) + \mu\int_a^b K(x, y)\psi(y)\,dy. \tag{12}$$

In this case or the kernal K(x, y) with s transform

$$K_s(\lambda, y) = \bar{s}(\lambda, y)f(\lambda), \tag{13}$$

the s transform of the unknown function is found algebraically:

$$\psi_s(\lambda) = \frac{\varphi_s(\lambda)}{1-\mu f(\lambda)}.$$

Examples of such equations are those with a kernel K(x-y) [1], those with a kernel which is expansible in products of Bessel functions [1], and the equation

$$\psi(x) = \varphi(x) + \mu\int_1^\infty \frac{\psi(y)}{x+y}\,dy,$$

which is solvable by means of a Mehler-Fock transform [6].

5. In the case when the upper or lower limits of integration are variable,

$$\varphi(x) = \int_a^x K(x, y)\psi(y)\,dy, \tag{14}$$

the Volterra equation (14) can be reduced to the Fredholm equation (1) with b = ∞ by the replacement $K(x, y)\to\theta(x-y)K(x, y)$, where θ(x) is the step function. Variability of the lower limit can be eliminated in an analogous way. The method developed above is applicable to many equations of the form (14). An example will be given below.

6. If the function $f(\lambda)$ is such that in solution (10) it is possible to change the order of integration, then Eq. (1) has the resolvent solution

$$\psi(x) = \int_c^d R(x, y)\varphi(y)\,dy, \tag{15}$$

where the inverse kernel to K(x, y)

$$R(x, y) = \int_k^l d\lambda\, f^{-1}(\lambda)t(\lambda, x)\bar{s}(\lambda, y)$$

is factored in the $\bar{t}(\lambda, x)$ expansion

$$R_{\bar{t}}(y, \lambda) = \bar{s}(\lambda, y) f^{-1}(\lambda).$$

Treated as an equation for $\varphi(x)$ with a known $\psi(x)$, relation (15) admits inversion with the aid of the relation

$$\varphi_s(\lambda) = f(\lambda)\psi_{\bar{t}}(\lambda),$$

where

$$\psi_{\bar{t}}(\lambda) = \int_a^b \bar{t}(\lambda, x)\psi(x)\,dx. \tag{16}$$

In what follows, it is assumed to be possible to expand the "arbitrary" function in the complete set $\bar{t}(\lambda, x)$

$$\psi(x) = \int_k^l \bar{t}(\lambda, x)\psi_{\bar{t}}(\lambda)\,d\lambda \tag{17}$$

with inversion (16) and the replacement $t \to \bar{t}$. Such an expansion is possible for many transformations (Fourier, Kontorovich-Lebedev, Mehler-Fock).

7. In the case when the limits of integration in formulas (2) and (4) coincide ($p = k$, $q = l$), as is the case for integral transformations with $p = k = 0$, $q = l = \infty$ (Fourier, Hankel, Kontorovich-Lebedev, Mellin), and expansion (17) is valid, one can switch the order of expansion of the kernel $K(x, y)$ and begin with an expansion in $\bar{t}(\lambda, y)$.

$$K(x, y) = \int_p^q K_{\bar{t}}(x, \lambda)\bar{t}(\lambda, y)\,d\lambda, \tag{18}$$

where

$$K_{\bar{t}}(\lambda, x) = \int_a^b t(\lambda, y)K(x, y)\,dy.$$

Upon factorization of the $\bar{t}$ transform of the kernel

$$K_{\bar{t}}(\lambda, x) = f(\lambda)s(\lambda, x) \tag{19}$$

we obtain once again a relation between the transforms of $\varphi(x)$ and $\psi(x)$ in the form (9). It is often more convenient to begin the expansion of the kernel with formula (18). We also note that the factorization of (19) of the transforms of kernel (7) arises when a delta-like singularity is present in the double transform of the kernel $K_{s,\bar{t}}(\lambda_1, \lambda_2)$ (compare the analogous case for the $K_{s,s}(\lambda_1, \lambda_2)$ transform for Fourier and Hankel transforms in Ref.[1]). In fact, under the condition $p = k$, $q = l$, the double s,t transform of the kernel in the form

$$K_{s,t}(\lambda_1, \lambda_2) = g(\lambda_1, \lambda_2)\delta(\lambda_1 - \lambda_2),$$

where

$$K_{s,t}(\lambda_1, \lambda_2) = \int_c^d dx \int_a^b K(x, y)\bar{s}(\lambda_1, x)t(\lambda_2, y)\,dy$$

and

$$K(x, y) = \int_k^l \int_k^l d\lambda_1\,d\lambda_2\,K_{s,t}(\lambda_1, \lambda_2)s(\lambda_1, x)\bar{t}(\lambda_2, y)$$

leads to Equation (11). Setting $g(\lambda, \lambda) = f(\lambda)$, we obtain factored transforms of kernels (7) or (19) after integrating over λ_1 or λ_2. Therefore, the entire method used in the derivation of solution (10) can be called the method of double integral transforms.

8. The evaluation of the integrals in formula (10) is simplified by having detailed tables of integral transforms, which give the intervals occurring in solution (10) for many types of integral transforms and functions $\varphi(x)$ [6-8].

An awkward but rather effective method of determining whether the kernel $K(x, y)$ is factorable, i.e., has s transforms (7) or the structure of Eq. (11), is the direct expansion of the kernel $K(x, y)$ in various kernels of integral transformations (5) (for this, y is treated as a parameter). The number of integral transformations which are usable in practice is not large (of the order of ten), and the tables of transforms are rather detailed and make it possible not only to check the factorability of the kernels, but also to construct several dozen substantially different factorable kernels. Examples of such kernels are given, in particular, by formulas 7.19, 7.20, 7.119, 8.26, 8.120, 9.143, 11.30, 11.138, 11.274, 11.300, 12.7, 12.14, and 12.18 of the book [6]. In order to clarify the way these are applied, we will look at some examples.

1. We will use the Kontorovich-Lebedev expansion of the kernel [6, formula 11.174]:

$$K(x, y) = \left(\frac{1}{2}\pi x\right)^{\frac{1}{2}} \exp(-xy), \tag{20}$$

$$K(x, y) = \int_0^\infty \lambda \operatorname{th}(\pi\lambda) P_{-\frac{1}{2}+i\lambda}(y) K_{i\lambda}(x)\, d\lambda, \tag{21}$$

where $K_\nu(x)$ is a modified Bessel function [9], which is real for imaginary index and $x > 0$; $P_\nu(x)$ is a spherical Legendre function of the first kind, which is also real for

$$\nu = -\frac{1}{2} + i\lambda;\ 0 \leqslant \lambda < \infty \text{ [9].}$$

Expansion of (21) shows that

$$K_{KL}(\lambda, y) = \lambda \operatorname{th}(\pi\lambda) P_{-\frac{1}{2}+i\lambda}(y)$$

contains as a factor the kernel of an inverse Mehler-Fock transformation $P_{-\frac{1}{2}+i\lambda}(y)$. Therefore, the transform of the Mehler-Fock expansion [6] of the unknown function $\psi(x)$

$$\psi_{MF}(\lambda) = \int_1^\infty dy\, \psi(y) P_{-\frac{1}{2}+i\lambda}(y)$$

is connected with the transform of the Kontorovich-Lebedev expansion [6]

$$\varphi_{KL}(\lambda) = 2\pi^{-2} \lambda \operatorname{sh} \pi\lambda \int_0^\infty dx\, \varphi(x) x^{-1} K_{i\lambda}(x),$$

$$\varphi(x) = \int_0^\infty K_{i\lambda}(x) \varphi_{KL}(\lambda)\, d\lambda \tag{22}$$

of the left-hand side of the equation

$$\varphi(x) = \int_1^\infty \left(\frac{1}{2}\pi x\right)^{\frac{1}{2}} e^{-xy} \psi(y)\, dy \tag{23}$$

by the simple relation

$$\psi_{MF}(\lambda) = \varphi_{KL}(\lambda)\,\lambda^{-1}\,(\mathrm{th}\,\pi\lambda)^{-1}. \qquad (24)$$

Inverting the MF transform (24) using the formula [6]

$$\psi(x) = \int_0^\infty d\lambda\lambda\,\mathrm{th}\,\pi\lambda\,\psi_{MF}(\lambda)\,P_{-\frac{1}{2}+i\lambda}(x)\,d\lambda,$$

we obtain a solution of Equation (23) in the form

$$\psi(x) = 2\pi^{-2}\int_0^\infty d\lambda\lambda\,\mathrm{sh}\,\lambda\,P_{-\frac{1}{2}+i\lambda}(x)\int_0^\infty dy\cdot y^{-1}\,\varphi(y)\,K_{i\lambda}(y). \qquad (25)$$

The reason that a simple Laplace transformation (kernel $e^{-\lambda x}$) cannot be applied to the function (20) is that the lower limit of integration in Eq. (23) is not zero, but unity.

2. The Weber integrals [10]

$$\int_0^\infty J_0(x\lambda)\cos(\lambda y)\,d\lambda = (x^2-y^2)^{-\frac{1}{2}}\,\theta(x-y), \qquad (26)$$

$$\int_0^\infty J_0(x\lambda)\sin\lambda y\,d\lambda = (y^2-x^2)^{-\frac{1}{2}}\,\theta(y-x) \qquad (27)$$

make it possible to solve, using the Hankel expansion in $J_0(\lambda x)$

$$\varphi(x) = \int_0^\infty d\lambda\lambda\,J_0(\lambda x)\,\varphi_{H0}(\lambda)$$

and the Fourier cosine or sine transform of the function $\psi(y)$, the Volterra equation

$$\varphi(x) = \int_0^x K(x, y)\,\psi(y)\,dy,$$

or

$$\varphi(x) = \int_x^\infty K(x, y)\,\psi(y)\,dy$$

with kernels (26) and (27) multiplied by arbitrary functions $\alpha(x)$ and $\beta(y)$.

3. The inhomogeneous Equation (12) ($a = 0$, $b = \infty$) with kernel

$$K(x, y) = (2y)^{-1}\exp\left[-\frac{\gamma}{2}\left(\frac{x}{y}+\frac{y}{x}+\frac{xy}{\gamma^2}\right)\right] \qquad (28)$$

can be solved by a Kontorovich-Lebedev [6] transform with the aid of the integral [8, v. II]

$$\int_0^\infty \lambda\,\mathrm{sh}\,\pi\lambda\,K_{i\lambda}(y)\,K_{i\lambda}(\gamma)\,d\lambda = \frac{\pi^2}{4}\exp\left[-\frac{\gamma}{2}\left(\frac{x}{y}+\frac{y}{x}+\frac{xy}{\gamma^2}\right)\right].$$

The KL transforms (22) of the functions $\psi(x)$ and $\varphi(x)$ are related by

$$\psi_{KL}(\lambda) = \varphi_{KL}(\lambda)/(1-\mu K_{i\lambda}(\gamma)). \qquad (29)$$

The inversion of this relation gives the solution of the inhomogeneous Equation (12) with kernel (28). We will not analyze here the applicability of formulas (25) and (29).

One can give a more systematic method of identifying kernels with factorable transforms (13), (19) for transform s(λ, x), t(λ, x) which are eigenfunctions of linear second-order differential operators acting on the variable x:

$$L_{sx} s(\lambda, x) = \pm \lambda^2 s(\lambda, x),$$
$$L_{tx} t(\lambda x) = \pm \lambda^2 t(\lambda, x). \qquad (30)$$

Writing the factored kernel (6) in the form

$$K(x, y) = \int_p^q s(\lambda, x) \bar{t}(\lambda, y) f(\lambda) d\lambda \qquad (31)$$

and operating on it with the operators (30), we obtain the relation

$$(L_{sx} \pm L_{\bar{t}y}) K(x, y) = 0, \qquad (32)$$

which is equivalent to the factorability condition of kernel (31). In formula (32) the minus sign is taken when the signs in the right-hand side of Eqs. (30) are the same. In Eq. (32) the integration over λ must be done after the addition (subtraction), the operators L_{sx} and $L_{\bar{t}y}$ having first brought under the integral sign. This method renders harmless the possible divergence of the integrals $\int_p^q \lambda^2 s(\lambda, x) \bar{t}(\lambda, y) f(\lambda) d\lambda$. If the operators L_{sx} are simple, it will be easier to check the factorability of the kernel K(x, y) using Eq. (32) than to expand the kernel in s(λ, x). Unfortunately, the operators are not simple for all transformations, and, furthermore, condition (32) is not satisfied for kernels of the form (11).

In many cases the differential operators look rather simple:

for the kernels of the Fourier representations (sin λx, cos λx, $e^{i\lambda x}$)

$$L_{Fx} = \frac{d^2}{dx^2}, \qquad (33)$$

the sign of the right-hand side of Eq. (3) is minus;

for the Laplace representation ($e^{-\lambda x}$) - operator (33), the sign is plus;

for the Hankel representation ($J_\nu(\lambda x)$ without the factor $x^{\frac{1}{2}}$)

$$L_{H\nu x} = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{\nu^2}{x^2}, \qquad (34)$$

the sign is minus;

the Mayer representation [6] (K (λx)) - operator (34), the sign is plus;

the Mellin representation ($x^{-\lambda}$)

$$L_{Mx} = x\frac{d}{dx} + x^2\frac{d^2}{dx^2}$$

the sign is minus.

A simple example of the application of the differential operators is provided by the equation

$$\varphi(x) = \int_0^\infty \psi(y) (x^2 + y^2)^{-\frac{1}{2}} dy. \qquad (35)$$

It is natural to look for a decomposition of the kernel into Fourier and Hankel integrals, i.e., to apply the operators L_{Fy} and $L_{H\nu x}$ to the kernel

$$K(x, y) = (x^2 + y^2)^{-\frac{1}{2}}.$$

A simple calculation shows that the index ν of the operator (34) is zero and

$$(L_{H0x} - L_{Fy}) K(x, y) = 0.$$

It follows from this and from the semi-infinite character of the integration interval in Eq. (35) that the expansion can be done in $J_0(\lambda x)$, $K_0(\lambda x)$ and $\sin \lambda y$, $\cos \lambda y$. Because $K(x, y)$ is finite for $y = 0$, $x > 0$, the expansion goes in $\cos \lambda y$. The singularity of the kernel at $x = 0$, $y = 0$ makes the expansion in $K_0(\lambda x)$ preferable.

Recovering the factored kernel

$$K(x, y) = 2\pi^{-1} \int_0^\infty \cos \lambda y \, K_0(\lambda x) \, d\lambda$$

by inversion of the Mayer expansion [6], we obtain [10]

$$\psi(x) = -2i\pi^{-2} \int_0^\infty d\lambda \lambda \cos \lambda x \int_{\delta - i\infty}^{\delta + i\infty} dz \, \varphi(z) \, z \, I_0(\lambda z), \tag{36}$$

where $\delta > 0$ ensures the convergence of the inside integral in the complex z plane, $I_0(x)$ is a modified Bessel function [9], and the inside integral is taken first. We will not analyze the class of functions $\varphi(x)$ which admit solution (36).

In the case of kernels containing parameters, such as,

$$K(x, y) = x^\alpha y^\beta (x^2 + y^2)^{-\mu},$$

the above method of checking the factorability of kernels enables one to find those values of the parameters for which the kernel factors and the functions into which it decomposes.

Thus, the method of double integral transforms examined in this paper makes it possible to find formal inversions of Fredholm equations of the first kind for a wide class of kernels not treated in the literature.

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Department of Quantum Theory