BORHERS' ALGEBRAIC FORMALISM FOR ARBITRARY-SPIN FIELDS

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The algebraic approach has one of the leading places amongst rigorous methods in modern quantum-field theory. The technique for constructing the dynamics of quantum-field models is based on networks of local algebras for the observables and the fields, which are von Neumann or C* algebras. On the other hand, in the initial stage such models are usually specified as their Whiteman fields, which are unrestricted operators $A(\hat{f})$ satisfying a system of Whiteman axioms. Therefore, in examining the model one encounters the standard problem of constructing a network of local algebras for a given system of Whiteman fields $A(\hat{f})$. It is possible to algebrize the Whiteman theory in two ways. We can consider the fields $A(f)$ themselves as algebraic objects and the correspondence $f \to A(f)$ can be interpreted as the representation of the basic-function topological algebra in the algebra $L(\mathscr{D})$ of all unbounded operators in Hilbert space $\mathscr{H}$, defined in a common dense region $\mathscr{D}$. This representation can be obtained by means of a canonical procedure from a positive functional (the Whiteman functional) in basic-function algebra, where the axioms in the Whiteman theory are initially introduced for that functional, while the canonical procedure ensures that they apply for the representation operator system $A(\hat{f})$. That approach is called the algebraic Borhers' approach, and the test-function algebra on which it is based is called a Borhers algebra. On the other hand, we may construct bounded functions from the operators $A(\hat{f})$. These bounded functions will generate certain von Neumann algebras, namely the field algebras $\mathscr{F}(O)$, and the next task then will consist in proving the axioms of relativistic quantum theory for these algebras. Each of these approaches has advantages and its own applications.

In this note and one following it, we consider the algebraic description of a system of quantized arbitrary-spin fields. First we consider the first of the above approaches, for which we construct an extension of Borhers' algebraic formalism to the case of arbitrary spin, which constitutes a system of Whiteman fields for the local network of unbounded-operator algebras. The construction of the local network of von Neumann field algebras will be considered in the next paper.

1. A Whiteman field of arbitrary spin [1] is the generalized function $A(\hat{f})$ in the space of multicomponent basic functions $\hat{f}=(f_1 \dots f_r)$, $f_i \in S(R^4)$, whose values belong to the set of unbounded operators in Hilbert space $\mathscr{H}$. The field spin is determined by the tensor transformation law of the basic functions $\hat{f}$ under Poincare transformations (a, Λ) of Minkowski space M. In the present case, the arbitrary (finite) spin is such that the law takes the form

$$\hat{\mathrm{f}}(x) \to \hat{\mathrm{f}}^{(a,\lambda)}(x);\quad (\hat{\mathrm{f}}^{(a,\lambda)}(x))_i = \sum_{j=1}^{r} V_{ji}(A^{-1}) f_i(\Lambda^{-1}(x-a)), \tag{1}$$

where A is an element in the universal covering SL(2, C) of the connected component $L^{\uparrow}_{+}$ of the Lorentz group uniquely defining $\Lambda \in L^{\uparrow}_{+}$ from

$$\Lambda_{\mu\nu}(A) = \frac{1}{2}\,\mathrm{Sp}\,(\sigma_\mu A \sigma_\nu A^*) \quad \left(\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right),\ \sigma_i \quad \text{are Pauli matrices}$$

and V(A) is a representation of SL(2, C) of finite dimensions r. The fields $A(\hat{f})$ and the conjugate ones $A^*(\hat{f})$ are specified in a common region $\mathscr{H}$ dense in $\mathscr{D}$ (the Gording region) and satisfy the standard system of Whiteman axioms. According to the relativistic covariance axiom, in $\mathscr{H}$ we are given a unitary strongly continuous representation $U(a\Lambda)$ of the Poincare group that leaves invariant the region $\mathscr{D}$ and the cyclic vacuum vector $\xi_0 \in \mathscr{D}$ and is such that

$$U(a\Lambda)\, A(\hat{\mathrm{f}})\, U(a\Lambda)^{-1} = A(\hat{\mathrm{f}}^{(a,\Lambda)}),$$

where $\hat{\mathrm{f}}(a, \Lambda)$ is given by (1). The fields $A(\hat{f})$ form a polynomial algebra $\mathscr{P}$ in $\mathscr{D}$, which for the sake of generality we will not assume to be irreducible. Also, we do not assume that the vacuum vector ξ_0 is unique.

2. As our Borhers algebra we consider the algebra Ω_0 of basic functions taking the form

$$\Omega_0 = \bigoplus_{n=0}^{\infty} (S(R^4) \oplus r)^{\otimes n}.$$

Any element $\hat{\mathrm{f}} \in \Omega_0$ is a sequence, in which term n

$$(\hat{\mathrm{f}})_n \equiv \hat{f}_n = \{f_{i_1,\ldots,i_n}(x_1, \ldots, x_n)\}_{i_1,\ldots,i_n=1}^{r}$$

is a tensor of group SL(2, C) that transforms as the irreducible product representation $\oplus V(A^{-1})$ of the n representations $V(A^{-1})$ of dimensions r; we assume that the number of nonzero terms in the sequence is always finite. By means of a natural generalization the scalar case enables us to define in Ω_0 the noncommutative product

$$(\hat{\mathrm{f}} \cdot \hat{\mathrm{g}})_n = \sum_{k=0}^{n} f_{i_1,\ldots,i_k}(x_1, \ldots, x_k)\, g_{i_{k+1},\ldots,i_n}(x_{k+1}, \ldots, x_n).$$

The Schwarz kernel topology from $S(R^4)$ is transferred to Ω_0 and converts Ω_0 into a topological algebra containing unit $\hat{\mathrm{e}} = (1, 0, 0, \ldots)$. Apart from equivalence, we define in Ω_0 the representation of the Poincare group $\mathscr{P}^{\uparrow}_{+}$:

$$\begin{gathered} \forall (a, \Lambda) \in \mathscr{P}^{\uparrow}_{+},\quad \forall \hat{\mathrm{f}} \in \Omega_0 \quad \hat{\mathrm{f}} \to \hat{\mathrm{f}}^{(a,\Lambda)} = (\hat{f}_n^{(a,\Lambda)})_{n=0}^{\infty}; \\ \hat{f}_n^{(a,\Lambda)}(x_1, \ldots, x_n) = \{f_{i_1,\ldots,i_n}^{(a,\Lambda)}(x_1, \ldots, x_n)\}_{i_1,\ldots,i_n=1}^{r} \end{gathered} \tag{2}$$

The term apart from equivalence is to be understood as follows: two elements of the algebra $\hat{f}$ and $\hat{g}$ having identical tensor structures transform as equivalent representations, but not necessarily identically equal ones.

It is readily seen that the representation is matched to multiplication in Ω_0:

$$(\hat{\mathbf{f}}\hat{\mathbf{g}})^{(a,\Lambda)} = \hat{\mathbf{f}}^{(a,\Lambda)}\hat{\mathbf{g}}^{(a,\Lambda)}. \tag{3}$$

The group SL(2, C) allows two forms of nonequivalent irreducible representations. Therefore, we can consider also the algebra Ω_0^η in which we are given the conjugate representation $V(A^*)$. The algebra Ω_0^η consists of elements of the following form:

$$(\hat{\mathbf{f}}^\eta)_n = \{\bigotimes_n \eta f_{i_n,\ldots,i_1}\}_{i_1,\ldots,i_n=1}^r,$$

where η is a matrix realization of the linear parity operator, i.e., the generator of the spatial reflection M. The parity operator has the following properties:

$$\eta^2 = E,$$

$$\bigotimes_n (\eta V(A^{*-1})\eta) = \bigotimes_n V(A).$$

Then $(\hat{\mathbf{f}}^\eta)_n$ transforms as $\otimes V(A^*)$. As the representations $V(A^{-1})$ and $V(A^*)$ are not equivalent, we cannot introduce involution in a natural fashion into Ω_0 (via the complex conjugate). Therefore, as our Borhers algebra we will in future use $\Omega_0^* = \Omega_0 \oplus \Omega_0^\eta$. In accordance with the generally accepted symbols $\mathscr{D}^{(mn)}$ and $\mathscr{D}^{(nm)}$ for irreducible nonequivalent representations of SL(2, C) we get that the representation $\mathscr{D}^{(mn)} \oplus \mathscr{D}^{(nm)}$ acts in Ω_0^*. Representations of this form are called real. As the description of real physical fields uses only real representations of SL(2, C) (see, for example, [2]), this constraint on V(A) is not a fundamental one.

Then in $\Omega_0^* = \Omega_0 \oplus \Omega_0^\eta$ we can introduce the involution

$$(\hat{\mathbf{f}}^+)_n = \{\bigotimes_n \eta f^+_{i_n,\ldots,i_1}\}_{i_1,\ldots,i_n}^r,$$

where the plus sign denotes the complex conjugate in application to the sequence terms.

3. The description of a physical system in this formalism is specified by means of a positive functional W in the algebra Ω_0^* subordinate to the system of axioms and called the Whiteman functional. The axioms imposed on W are obtained by direct reformulation of the Whiteman standard axioms in terms of the algebra Ω_0^* and functional W.

We formulate the relativistic covariance axiom in the usual way:

$$W(\hat{\mathbf{f}}^{(a,\Lambda)}) = W(\hat{\mathbf{f}}) \quad \forall \hat{\mathbf{f}} \in \Omega_0^*, \ (a, \Lambda) \in \mathscr{P}_+^\uparrow. \tag{4}$$

The formulation for the locally axiom in the Borhers formalism is constructed in such a way as to provide local commutativeness in the operators for the representation Π corresponding to W. This leads to the requirement that the null space J_0 of functional W includes the locality ideal J_l, namely, the two-sided ideal of Ω_0^* generated by linear combinations of the products of the elements $\hat{f}$ such that

$$\hat{f}_n(x_1, \ldots, x_n) = g(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) -$$
$$- g(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n),$$

where $g(x_1, \ldots, x_n) = 0$ for $(x_i - x_{i+1})^2 \geqslant 0$. In our case it must be borne in mind that these fields may also be anticommutating. The commutation structure may be expressed as a unique condition called locality with torsion in the general system of fields including both commutating and anticommutating operators. This condition is formulated by means of a special transformation upon the set of fields, which acts identically on the commutating (boson) operators and changes the signs of the anticommutating (fermion) operators. In order that the operators of representation Π should satisfy locality with torsion, we clearly have to define the transformation of torsion in the algebra Ω_0^*, which has a unitary realization in the space of representation Π. Further, on the basis of this transformation it is necessary to modify the definition of the locality ideal J_l. It has been shown [3] that the torsion operation can be constructed if we are given the representation of the Poincare group. Then the desired operation is defined via a transformation corresponding to the element (0; -I) from the universal covering $\bar{\mathcal{P}} = (M, SL(2, C))$ of the Poincare group.

Following [3], we consider in Ω_0^* an automorphism u_0 corresponding to (0; -I):

$$u_0 : \hat{\mathbf{f}} \to \hat{\mathbf{f}}^{(0,-I)} \quad \forall \hat{\mathbf{f}} \in \Omega_0^*$$

and define the automorphism of torsion z in Ω_0^* by

$$z = (1+i)^{-1}(1+iu_0);\ \hat{\mathbf{f}} \to \hat{\mathbf{f}}^z \equiv (1+i)^{-1}(\hat{f}^{(0;-I)})\ \forall \hat{\mathbf{f}} \in \Omega_0^*.$$

To derive the necessary modification of the locality ideal, we note that in the scalar case J_l can be put as

$$J_l = \Big\{ \hat{\mathbf{g}} \in \Omega_0^* \,|\, \hat{\mathbf{g}} = \sum_{k=1}^{N} \hat{\mathbf{c}}_k (\hat{\mathbf{a}}_k \hat{\mathbf{b}}_k - \hat{\mathbf{b}}_k \hat{\mathbf{a}}_k) \hat{\mathbf{d}}_k;\ \hat{\mathbf{c}}_k,\ \hat{\mathbf{d}}_k \in \Omega_0^*;\ \hat{\mathbf{a}}_k,\ \hat{\mathbf{b}}_k \in \Omega_0^{*(1)},$$
$$\operatorname{supp} \hat{\mathbf{a}}_k \sim \operatorname{supp} \hat{\mathbf{b}}_k,\ N = 1, 2, \dots \Big\},$$

where by definition

$$\Omega_0^{*(1)} \overset{\text{def}}{=} \{ \hat{\mathbf{f}} \in \Omega_0^* \,|\, \hat{f}_n = 0,\ n \geqslant 2 \}$$

and the symbol ~ denotes mutual space-likeness. From this representation we determine the ideal $J_{l,z}$ of locality with torsion in the Borhers' algebra by the following formula:

$$J_{l,z} = \Big\{ \hat{\mathbf{g}} \in \Omega_0^* \,|\, \hat{\mathbf{g}} = \sum_{k=1}^{N} \hat{\mathbf{c}}_k (\hat{\mathbf{a}}_z \hat{\mathbf{b}}_k^z - \hat{\mathbf{b}}_k^z \hat{\mathbf{a}}_k) \hat{\mathbf{d}}_k;\ \hat{\mathbf{c}}_k,\ \hat{\mathbf{d}}_k \in \Omega_0^*;$$
$$\hat{\mathbf{a}}_k,\ \hat{\mathbf{b}}_k \in \Omega_0^{*(1)},\ \operatorname{supp} \hat{\mathbf{a}}_k \sim \operatorname{supp} \hat{\mathbf{b}}_k,\ N = 1, 2, \dots \Big\}. \tag{5}$$

Correspondingly, we formulate as follows the condition for generalized local commutativeness (locality with torsion):

$$J_{l,z} \subset J_0. \tag{6}$$

Finally, the formulation of the spectral condition does not require any changes: as in the scalar theory, we introduce the spectrality ideal, namely the right ideal

$$J_{sp} = \{ \hat{\mathbf{f}} \in \Omega_0^* \,|\, f_0 = 0,\ f_{i_1,\dots,i_n}(p_1, \dots, p_n) = 0 \}$$

in the region of the cone

$$p_1 \in \bar{V}^+, \ldots, p_1 + \ldots + p_n \in \bar{V}^+$$

and impose the requirement $J_{sp} \subset J_0$.

4. The functional W canonically defines the representation Π of the algebra Ω_0* in the Hilbert space $\mathscr{H}$, obtained as a completion of the factor space $\Omega = \Omega_0^*/J_0$ on the norm generated by the scalar product in Ω.

$$(\xi(\hat{f}), \xi(\hat{g})) = W(\hat{g}\hat{f}^*) \quad \forall \hat{f}, \forall \hat{g}.$$

The representation operators are unbounded closable operators in $\mathscr{H}$ having the compact region Ω and defined by

$$\Pi(\hat{f})\xi(\hat{g}) \equiv \xi(\hat{g}\hat{f}) \quad \forall \hat{f} \in \Omega_0^* \quad \forall \xi(\hat{g}) \in \Omega.$$

The transform of the representation $\Pi(\Omega_0^*) \equiv \mathscr{P}$ is an Op* algebra in the region and the involution $\Pi(\hat{f}) \to \Pi(\hat{f})^* \equiv \Pi^*(\hat{f})|_\Omega$ together with the cyclic vector $\xi_0 = \xi(e)$. It is readily checked that the operators $\Pi(\hat{f})$ satisfy the system of Whiteman axioms.

Relativistic covariance. It follows from (3) and (4) that the ideal J_0 is Poincare invariant: $J_0^{(a,\Lambda)} = J_0$ for all $(a, \Lambda) \in \mathscr{P}_+^\uparrow$. Therefore, the representation Π is Π $\mathscr{P}_+^\uparrow$-covariant, i.e., in $\mathscr{H}$ we have unambiguously defined operators U(a, Λ) that provide the representation of $\mathscr{P}_+^\uparrow$: :

$$U(a, \Lambda)\xi(\hat{f}) = \xi(\hat{f}^{(a,\Lambda)}), \quad \forall (a, \Lambda) \in \mathscr{P}_+^\uparrow, \quad \hat{f} \in \Omega_0^*.$$

On account of (4), all the U(a, Λ) are unitary, and the cyclic vector ξ_0 (vacuum) is Poincare-invariant.

Locality. As a consequence of the relativistic covariance, the automorphism z of algebra Ω_0* is realized in representation Π by a unitary operator Z such that

$$\forall \hat{f} \in \Omega_0^* \quad Z\xi(\hat{f}) = \xi((1+i)^{-1}(\hat{f} + i\hat{f}^{(0; -I)})).$$

The operator Z provides a torsion transformation on the field operators $\Pi(\hat{f})$:

$$\Pi(\hat{f}) \to \Pi(\hat{f})^z = Z\Pi(\hat{f})Z^{-1},$$

which is an automorphism of the Op* algebra $\mathscr{P}$. From (5) one readily gets that condition (6) for the Whiteman functional leads to obedience to the axion of locality with torsion:

$$[\Pi(\hat{f}), \Pi(\hat{g})^z] = 0 \quad \forall \hat{f}, \hat{g} \in \Omega_0^{*(1)} \quad \operatorname{supp}\hat{f} \sim \operatorname{supp}\hat{g},$$

and equality is understood upon region Ω.

Spectrality. By analogy with the scalar theory, the condition for the Whiteman functional leads to the standard formulation of spectrality for the representation U(a, Λ):

$$U(a, I) = \int e^{ipa}\, dE(p), \ \operatorname{supp} E(p) \subset \bar{V}_+.$$

Therefore, the system of arbitrary-spin fields can be described in a compact fashion by means of a positive functional in Borhers' algebra, and this description is completely equivalent to the Whiteman one.

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