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THE QUANTUM HALL EFFECT AND THE GENERAL PRINCIPLES OF QUANTUM FIELD THEORY

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The quantum Hall effect (QHE) is considered from the standpoint of the general theorems of quantum field theory and is interpreted as a quantum macroscopic phenomenon due to a spontaneous break of symmetry of the corresponding quantum system. An effective field model of QHE is suggested and a relationship between this model and the two-dimensional Higgs abelian model is established. This made it possible to determine what seems to be the true vacuum state in the case of the fractional quantum Hall effect, which turned out to be a nondegenerate $|\theta>$-vacuum state.

1. The quantum Hall effect (QHE), a macroscopic phenomenon in the two-dimensional electron gas in a quantizing magnetic field, is the most interesting and remarkable discovery in the end of the 20th century [1]. At the present time distinction is made between the integral and fractional (or anomalous) quantum Hall effects (IQHE and FQHE, respectively). The IQHE is brought about by a spontaneous break of translational invariance of the quantum system, although it can also be explained by studying, using methods of statistical physics, the behavior of a nearly degenerate ideal two-dimensional electron gas in a quantizing magnetic field. The two most important properties of the IQHE, namely the independence of the Hall conductivity of the concentration of movable carriers and the existence of a plateau in the graph of the off-diagonal conductivity in an interval of the magnetic field strength variation, can be proved directly by determining the electron gas concentration as a function of the magnetic field in the state of thermodynamic equilibrium in the limit $\varkappa T \ll \hbar\omega$ (where $\varkappa$ is Boltzmann's constant; T is the gas temperature; $\omega = eH/(mc)$ is the cyclotron frequency; H is the magnetic field strength; e and m are the electron charge and mass; and c is the velocity of light) [2]. The IQHE arises in the state of thermodynamic equilibrium and is a dissipation-free effect, and, conversely, the absence of dissipation is an indication of the thermodynamic equilibrium of the electron gas.

From the standpoint of quantum field theory (QFT) it is possible to relate these two properties to the principle of gauge invariance of QFT and to show that the two IQHE properties are generated by the so-called discontinuous gauge transformations of the vector potential [2].

In contrast to the IQHE, the fractional quantum Hall effect arises in much stronger magnetic fields, and in this case the electron gas cannot be considered ideal. Qualitatively, this can be demonstrated in the following way. Consider N electrons moving in the xy plane (on an area L^2) in a magnetic field $\mathbf{H} = (0, 0, H)$. The surface electron density is $\sigma = N/L^2$, and $\pi r_0^2 = \sigma^{-1}$ is the area per electron. In a real situation the total electron charge $-Ne$ is neutralized by the positive background charge Ne (which is assumed to be homogeneous), so that the electron interaction is mainly due to the exchange effects. The condition for the idealness of the gas is $E_F > U_{\rm int}$, where E_F is the Fermi energy. On substituting the values of E_F and $U_{\rm int}$ found in [3] into this condition we obtain

$$\sigma > \frac{2\sqrt{2}}{3\pi a_B a}. \tag{1}$$

Here $a_B = \hbar^2/me^2$ is the Bohr radius and $a = \sqrt{\hbar c/eH}$ is the "magnetic length".

The electron energy levels in the magnetic field (the Landau levels) are

$$E_n = \hbar\omega(n + 1/2), \tag{2}$$

and to each value of the quantum number n there correspond

$$\Delta = eHL^2/(2\pi\hbar c) \tag{3}$$

states; Δ is the multiplicity of degeneration of the Landau level. For an integral value of ν the ratio $N/\Delta = \nu$ is equal to the number of totally filled levels. If $\nu < 1$, then ν is interpreted as the occupancy of the lowest Landau level for the given N and H. The factor ν is related to the quantum number l characterizing the projection of the orbital momentum of relative motion of an electron pair onto the field direction [4, 5].

According to [4-6], the distances between pairs of noninteracting electrons in the ground state are $\rho^2 = 2la^2$, $l \neq 0$, and, since it is assumed that in the ground state the spins of all electrons are oriented opposite to the field, the number l is odd. The inclusion of the Coulomb interaction does not change these results since the spectrum of the operator $U(|r_i - r_j|)$ for functions belonging to the same Landau level is quantized [6, p. 289]. It is also clear that for a homogeneous background charge and a homogeneous magnetic field the spatial distribution of electrons must be homogeneous. Therefore for any pairs we have $r_0^2 = \rho^2$, that is

$$N/\Delta = 1/l. \tag{4}$$

The FQHE theory for the case $\nu = 1/3$ was constructed by Laughlin [4-6]. According to the theory, the FQHE is due to the condensation of electrons in the ground state, which is an incompressible quantum Fermi liquid with elementary excitations possessing a fractional electric charge (equal to $e\nu$). This conclusion of Laughlin appears to be very important since it apparently allows one to prove the existence of the FQHE on the basis of general QFT theorems (the theorems on spontaneous break of symmetry in a quantum system). Indeed, the second-quantized Hamiltonian of the electron system in question includes the electron birth and annihilation operators that are transformed under a gauge transformation like

$$\widehat{\psi} \to \widehat{\psi} \exp\left\{\frac{ief}{c\hbar}\right\}, \tag{5}$$

while, owing to the electron condensation, the quasi-particles carry the charge $e\nu$, and, consequently, the wave function of the Hamiltonian ground state is transformed in a different way in the second-quantized theory. It should also be noted here that the total Hamiltonian of the system is invariant relative to transformations (5) whereas the vacuum is not. The noninvariance of vacuum results in the appearance of noninvariant physical quantities (the macroscopic current in the FQHE regime in the case under consideration) whose variation is described in the language of the classical theory (here "classical" means non-second-quantized).

In quantum field theory one distinguishes between spontaneous breaks of the global and of the local symmetry in the theory. For instance, the famous Goldstone theorem holds [6, p. 341] that a spontaneous break of the continuous global symmetry results in the appearance of a massless spinless particle (the Goldstone excitation). There are well-known examples of such Goldstone excitations in nonrelativistic many-body theories, such as spin waves in ferromagnetics and phonons in crystals.

With a spontaneous break of the local gauge symmetry no Goldstone mode appears, but instead the gauge field becomes massive. The nongauge field $\varphi(x)$ producing a symmetry break is called the Higgs field. Here the QFT effects also have similarity with the effects in the nonrelativistic many-body theory. For example, in a superconductor this leads to the appearance of a photon mass and, ultimately, to the Meissner effect.

2. Although the FQHE is a many-particle quantum mechanics phenomenon, it appears important to formulate an effective field theory for it similar to the remarkable Ginzburg-Landau theory of superconductivity. Such a theory was elaborated by Girvin [6, pp. 341 and 368], who indicated that the theory must take into account the following three fundamental FQHE features: (i) the phonon excitations in the FQHE have no long-wave asymptotics, i.e., the phonons in the FQHE have a mass; (ii) in the FQHE theory there are isolated eddying solutions with finite energy; and (iii) the eddies in the FQHE are electrically charged, their charge being equal to $e\nu$.

It is assumed that in superconductors the features (i) and (ii) manifest themselves via the Anderson-Higgs mechanism [7, 8], but without the fundamental Higgs field. As is well known, the condensation phenomenon and the interaction of charged Cooper pairs with the (self-generated) vector potential, owing to which the "Higgs mechanism" appears, are described phenomenologically in the Ginzburg-Landau theory by introducing a complex "ordering parameter" $\varphi(x)$, the expression $|\varphi(x)|^2$ being the local particle density. In the case of the FQHE the situation with the "Higgs mechanism" looks less convincing since the currents in the FQHE regime generate a negligibly small magnetic flux [6, pp. 341 and 368], which leads to difficulties in the interpretation of quantities of the type of the vector potential that are necessarily involved in the theory.

Nevertheless we shall consider electrons in an external magnetic field $\mathbf{H} = (0, 0, H)$ assuming that they are in motion in the xy plane. The electron field will be described by a complex scalar function $\varphi(r, t)$; this corresponds to the case of electrons filling only the lowest Landau level (the electron spins having the same orientation). The vector potential of the external magnetic field is taken in the form

$$\mathbf{A}_{\text{ext}} = (0, xH, 0). \tag{6}$$

Then the vector potential of the total electromagnetic field is represented in the following way:

$$\widetilde{\mathbf{A}} = \mathbf{A}_{\text{ext}} + \mathbf{A}. \tag{7}$$

Here the second term describes the potential generated by the electron current. This splitting seems reasonable since $\mathbf{A}$ refers to the field φ which includes the action of the external field. Representation (7) also means that even at the classical level we have already partly fixed the gauging of the 4-potential A_μ by means of the condition $A_0 = 0$.

The density of the model Lagrangian has the form

$$\mathcal{L} = \frac{1}{2m}\left[\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\widetilde{\mathbf{A}}\right)\varphi\right]^*\left[\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\widetilde{\mathbf{A}}\right)\varphi\right] + \frac{\hbar}{i}\frac{\partial\varphi^*}{\partial t}\frac{\partial\varphi}{\partial t} - \frac{\mathbf{B}^2}{8\pi}. \tag{8}$$

Here the operator $\nabla = (\partial/\partial x, \partial/\partial y)$, and $\mathbf{B}$ is the total magnetic field. Note that expression (8) is invariant with respect to the local gauge transformations

$$\mathbf{A} \to \mathbf{A} + \nabla\omega. \tag{9}$$

Applying the variational principle one can derive from (8) the Schrödinger equation and the Maxwell equations for the magnetic field $\mathbf{B}$. Setting $\mathbf{A} = 0$ in the Schrödinger equation we arrive at the well-known problem on electron motion in a constant magnetic field in the xy plane; the eigenfunctions of the Hamiltonian of such a system have the form [9]

$$\varphi_n = \exp\left\{\frac{ipy}{\hbar}\right\}\chi_n(\eta), \quad \chi_n(\eta) = \frac{\exp\left\{-\frac{\eta^2}{2}\right\}H_n(\eta)}{\pi^{1/4}a^{1/2}\sqrt{2^n n!}}, \tag{10}$$

where $\eta = (x - x_0)/a$, n is the index of the Landau level, p is an eigenvalue of the operator $(\hbar/i)\nabla_y$, and $x_0 = cp/(eH)$ is the classical x coordinate of the electron orbit center. It can easily be shown that the unperturbed (by the field $\mathbf{A}$) ground state φ_0 is eliminated by the following operator:

$$(\partial/\partial\eta + \eta)\varphi_0 = 0. \tag{11}$$

We shall require that (11) should also hold for the ground state function ψ in the self-consistent problem, i.e., with consideration for $\mathbf{A}$. For $\mathbf{A} = 0$, the Maxwell equations

$$\operatorname{rot}\mathbf{B} = -\frac{4\pi}{c}\mathbf{j}, \quad \mathbf{j} = \frac{e}{2m}\left\{\psi^*\left(\frac{\hbar}{i}\nabla - \frac{e}{c}\widetilde{\mathbf{A}}\right)\psi + \psi\left(-\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}\right)\psi^*\right\} \tag{12}$$

imply that

$$\frac{\partial B_z}{\partial x} = -\frac{4\pi}{c}j_y = -\frac{4\pi}{c}\frac{e\hbar}{ma}\eta|\psi|^2, \tag{13}$$

and, by virtue of (11), we obtain

$$\frac{1}{a}\frac{\partial}{\partial\eta}\left(B_z + \frac{2\pi e\hbar}{mc}|\psi|^2\right) = 0, \tag{14}$$

whence

$$B_z = H - \frac{2\pi e\hbar}{mc}|\psi|^2. \tag{15}$$

Relations of the type of (11) can also be obtained by choosing the symmetric gauging of the vector potential:

$$\mathbf{A} = \frac{1}{2}[\mathbf{Hr}]. \tag{16}$$

In this case the unperturbed solution is written in the polar coordinates r, θ in the form

$$\psi \sim \exp\{il\theta\} I_{ns}(\rho), \qquad \rho = \frac{eH}{2c\hbar}r^2,$$

where l is the projection of the orbital momentum onto the z axis; $s > 0$ and $n = l + s$ are, respectively, the radial and principal quantum numbers; and $I_{ns}(\rho)$ is the generalized Laguerre polynomial. For $I_{ns}(\rho)$ the recurrence relation [10]

$$\sqrt{\rho}\left(2\frac{d}{d\rho} + 1 + \frac{l}{\rho}\right) I_{ns} = \sqrt{4n} I_{n-1s} \tag{17}$$

holds, whence, setting $n = 0$, we derive an analog of relation (11):

$$\sqrt{\rho}\left(\frac{l}{\rho} + 1\right) I_{0s} = 2\sqrt{\rho}\frac{dI_{0s}}{d\rho}. \tag{18}$$

Repeating the above argument we arrive at the relation

$$\frac{d}{dr}B_z = -\frac{4\pi}{c}j_\theta = -\frac{2\pi e\hbar}{mc}\frac{d}{dr}|\psi|^2 \tag{19}$$

instead of (13), and, further, at (16).

Let us consider static solutions. For these solutions the Hamiltonian of the system, which is an analog of the free energy functional in the Ginzburg-Landau theory, has the form

$$\mathcal{H} = \int d^2\mathbf{r}\left(\frac{\mathbf{B}^2}{8\pi} + \frac{1}{2m}\left|\left(-i\hbar\frac{\partial}{\partial \mathbf{r}} - \frac{e}{c}\mathbf{A}\right)\psi\right|^2\right). \tag{20}$$

We shall seek the solution $\psi(x, y)$ assuming that the dependence of $\psi(x, y)$ on y is determined by expression (10). Then (20) takes the form

$$\mathcal{H} = L_y \int d\eta \left\{\frac{1}{2m}|\mathbf{D}\psi|^2 + \frac{H^2}{8\pi}\left(1 - \frac{2\pi e\hbar}{mcH}|\psi|^2\right)^2\right\}.$$

Here $\mathbf{D} = \left(-\frac{i\hbar}{2a}\frac{\partial}{\partial\eta} - \frac{e}{c}A_\eta,\ -\frac{\hbar}{2a}\frac{\partial}{\partial\eta}\right)$ and L_y is the length of the region of motion in the direction of y. Denoting $\eta = x$ and introducing

$$\psi = \frac{\xi(x)}{\sqrt{\lambda a^2}}\exp\{i\alpha(x)\}, \qquad \lambda = 2\pi\frac{e^2}{mc^2}, \tag{21}$$

we obtain the Hamiltonian of the system in the form

$$\mathcal{H} = \frac{\hbar^2 L_y}{2m\lambda a^3}\int dx \left\{\left|\frac{d\xi}{dx}\right|^2 + \left|\left(\frac{\partial}{\partial x} - \frac{iea}{c\hbar}A_x\right)\xi\exp\{i\alpha(x)\}\right|^2 + (\xi^2 - 1)^2\right\}. \tag{22}$$

In the symmetric gauging of the vector potential (16) the Hamiltonian retains the form (22) with change of variables $L_y \to 2\pi a$, $x \to r/a$, $dx \to ardr$.

Of interest may be the solutions $\varphi(x)$, $A(x)$ with finite energy. For them each of the positive definite terms in Hamiltonian (22) must be finite. The minimum of the Hamiltonian corresponds to the ground state of the system. The configuration of the fields $\psi(x)$ and $A(x)$ for which the Hamiltonian attains minimum describes the "classical vacuum states" of the system. An appropriate quantum-field vacuum state must be constructed around them.

A relativistic analog of the theory described by Hamiltonian (22) is the Higgs abelian model in the $1+1$ dimensions which includes the electromagnetic field $A_\mu(x,\,t)$ interacting with the complex scalar field $\Phi(x,\,t)$ [11]. The model Lagrangian has the form [11]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\gamma}F^{\mu\gamma} + \frac{1}{2}|D_\mu\Phi|^2 - \frac{1}{4}\lambda'(|\Phi|^2 - F^2)^2, \tag{23}$$

where

$$F_{\mu\gamma} = \partial_\mu A_\gamma - \partial_\gamma A_\mu, \quad D_\mu\Phi = (\partial_\mu - ieA_\mu)\Phi, \quad \hbar = c = 1, \tag{24}$$

and F and λ' are real constants.

A remarkable feature of the Higgs model in the $1+1$ dimensions consists in the absence of the Higgs phenomenon owing to the inclusion of non-perturbative mechanisms by instanton effects, which in the $1+1$ dimensions are known [11] to differ from those in the $3+1$ dimensions.

One of the classical vacuum states of Hamiltonian (22) seems to be described by the solutions $\xi(x) = \pm 1$, $A_x = 0$. The nontrivial static solutions $A_x = 0$, $\alpha(x) = 0$, $\xi(x) = \pm\tanh[(x - x_0)/\sqrt{2}]$ are well known and are the "kink" (+) and the "antikink" (−). These solutions satisfy the corresponding boundary conditions $\xi(x)|_{x\to\infty} = \pm 1$.

When the symmetric gauging (16) is used for $A_r = 0$ and $\alpha(r) = 0$, the Hamiltonian has the form

$$\mathcal{H} = \frac{2\pi\hbar^2}{m\lambda a^2}\int r\,dr\left\{\left(\frac{d\xi}{dr}\right)^2 + \frac{1}{2}(\xi^2 - 1)^2\right\}. \tag{25}$$

It is obvious that the solutions $\xi(r) = \pm 1$ describe the absolute classical vacuum; the other nontrivial static solutions satisfy the equation

$$\frac{d^2\xi}{dr^2} + \frac{1}{r}\frac{d\xi}{dr} + \xi - \xi^3 = 0. \tag{26}$$

Here the solution $\xi(r)$ has the meaning of a real radial function that describes eddying states (analogs of Abrikosov eddies in superconductors), and we have $\xi(r)|_{r\to\infty} = \pm 1$. The eddy located at the origin*) is described by the equation

$$\psi(r,\,\theta) = f(r)\exp\{il\theta\}, \tag{27}$$

where θ is the azimuthal angle and l is the integral quantum number of the eddy. As is well known, the requirement that the energy of two-dimensional eddy states should be finite leads to the quantization of the magnetic flux; this effect is related to the topological properties of eddying solutions. It turns out [6, p. 368] that in the FQHE regime the magnitude of the magnetic flux is determined by the electric charge related to the field ψ and not to the electric current, which is the case in superconductivity.

Indeed, consider a contour C around the eddy at large distances from it. In view of (21), formula (27) implies the following topological quantization condition:

$$\oint_C d\mathbf{r}\psi^*\left(-i\frac{\partial}{\partial\mathbf{r}}\right)\psi = 2\pi l, \tag{28}$$

and, by the continuity of ψ, it follows that l is an integer. On the other hand, the condition of finiteness of the eddy energy implies that the current density (12) tends to zero sufficiently fast at large distances, i.e.,

*) Equation (26) possesses the trivial solutions $\xi(r) = \pm 1$ and the asymptotics $\xi(r)|_{r\to\infty} = \pm 1$. Owing to this the kinetic energy density of the eddy is divergent at the origin due to the presence of a centrifugal barrier. To remove the divergence a "normal core" is assigned to the eddy by equating the radial function in the vicinity of $r = 0$ to $f(r) = 1 - \xi(r)$ or $f(r) = -(1 + \xi(r))$ for the solutions satisfying for $r \to \infty$ the boundary conditions $\xi(r)|_{r\to\infty} = 1$ or $\xi(r)|_{r\to\infty} = -1$, respectively.

$\oint_C \mathbf{j}d\mathbf{r} = 0$, and consequently

$$\oint_C d\mathbf{r}\psi^* \left(-i\frac{\partial}{\partial \mathbf{r}}\right)\psi = 2\pi l = -\frac{e}{c\hbar}\oint \tilde{A}d\mathbf{r},$$

whence, applying Stokes' formula, we obtain

$$2\pi l = -\frac{e}{c\hbar}\int_S B_z d^2\mathbf{r} = -\frac{eH}{c\hbar}\int_S d^2\mathbf{r}(1-\xi^2(r)). \tag{29}$$

Since HS is the external magnetic flux, the expression

$$q = \frac{e}{lS}\int d^2r(1-\xi^2(r)) \tag{30}$$

is the (fractional) charge of the eddy. The eddy charge is determined by the flux of the magnetic induction vector B across an area containing the entire eddy. According to the homotopic classification, l is the number of windings or the number of revolutions of the field phase $\alpha(x)$ in a circle at spatial infinity. It is well known [11] that for eddying solutions one can define the corresponding "topological charges" ("indices"), which are boundary conditions and are preserved owing to the finiteness of the energy. Using them one can distribute the solutions over some topological sectors. In the case under consideration the topological classification is particularly simple for the solutions $\psi(x)$. The topological charge can be defined for them as

$$Q = \xi(x=\infty) - \xi(x=-\infty) = 2 \tag{31}$$

and can be related to the corresponding preserved current

$$J^\mu = \varepsilon^{\mu\gamma}\partial_\gamma\xi, \qquad g_{\mu\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{32}$$

where $\varepsilon_{\mu\gamma}$ is an antisymmetric tensor. It follows from (32) that

$$\partial_\mu J^\mu = 0, \qquad Q = \int_{-\infty}^{\infty} dx\, J_0, \tag{33}$$

whence we again arrive at the expression

$$Q = \int_{-\infty}^{\infty} \frac{dx}{\cosh^2\frac{x}{\sqrt{2}}} = \tanh x \Big|_{-\infty}^{\infty} = 2. \tag{34}$$

The flux of the magnetic induction vector is

$$H\int_S d^2r(1-\xi^2(x)) = HL_y \int_{-L_x/2}^{L_x/2} dx\, \frac{1}{\cosh^2\frac{x}{\sqrt{2}}} = SHQ. \tag{35}$$

Here we have taken into account that a circle is topologically equivalent to a rectangle and that the dimension of the region of localization of the "kink" energy is $\sqrt{2}\,l_x$, where l_x is the unit length. As is seen from (30) and (35), the eddy charge can also be related to the topological charge.

Since the topological charge is nonzero, it follows that the potential $U(\xi)$ has two degenerate minima: $\xi = 1$ and $\xi = -1$. The minima are not invariant separately although the Hamiltonian is invariant with respect to the transformation $\xi \leftrightarrow -\xi$, i. e., the discrete symmetry of the classical theory is spontaneously broken for the $\xi(x)$ field. In the quantized theory this break must result in a macroeffect. In the case under consideration a spontaneous break of the discrete symmetry (SSB) manifests itself in the existence of a classical field $\xi(x)$ or $\xi(r)$, which can be represented as the vacuum quantum mean $\langle 0|\hat{\xi}|0\rangle = \xi(r)$. Consequently, the very fact of existence of solitons $\xi(x)$ or eddies with fractional charges turns out to be related to SSB. Note that these solutions are stable by virtue of the topological properties (conservation of the topological charge).

3. We now come back to the discussion of the FQHE properties following from the analogy with the relativistic Higgs abelian model. It is necessary to repeat and recall some results of that model; we shall follow [11]. Let us identify the configurations of the classical vacuum states of the model described by the Lagrangian (24); the minimum energy of the system is obviously attained provided that

$$\Phi(x, t) = F \exp\{i\alpha(x, t)\}, \qquad \mathcal{D}_\mu \Phi(x, t) = 0 \equiv F_{\mu\gamma}(x, t), \tag{36}$$

which is fulfilled for

$$A_\mu(x, t) = \frac{1}{ie\Phi}\partial_\mu \Phi = \frac{1}{ie}\exp\{-i\alpha(x, t)\}\partial_\mu \exp\{i\alpha(x, t)\}. \tag{37}$$

The dependence on time in (37) is completely determined by the time-dependent general gauge freedom of the theory (in fact, by the Lagrangian invariance with respect to the $U(1)$ gauge transformations) and can therefore be removed by means of partial gauge fixing $A_0(x, t) = 0$. In this gauging the classical vacuum states (36) and (37) no longer depend on time:

$$\Phi(x) = F \exp\{i\alpha(x)\}, \qquad A_x = \frac{1}{e}\frac{d\alpha(x)}{dx}. \tag{38}$$

Note that in the $1+1$ dimensions, for $A_0 = 0$ the 2-vector A_μ has only a single component A_x.

The gauging freedom of the theory is not completely fixed by the condition $A_0 = 0$ (there remain admissible time-independent gauge transformations), which in the context of quantum field theory leads to the well-known condition in the form of the Gauss operator law: $J(x) = dE_x/dx - j_0 = 0$ imposed on the physical states as the relation

$$J(x)|\Psi\rangle_{\text{ph}} = 0. \tag{39}$$

Further, the operator

$$U_\Lambda = \exp\left\{ \frac{i}{e}\int_{-\infty}^{\infty} \Lambda(x) J(x)\, dx \right\}$$

is introduced, where $\Lambda(x)$ is an arbitrary c-number function so that

$$U_\Lambda|\Psi\rangle_{\text{ph}} = |\Psi\rangle_{\text{ph}} \tag{40}$$

and small gauge transformations with $\tilde{\Lambda}(\mp\infty) = 0$ are defined, relative to which the physical states must be invariant. The gauge transformations that are not small are said to be large; only the Lagrangian and the Hamiltonian are invariant with respect to them.

The physical states are constructed as superpositions of eigenstates of the field operators $|\Phi, A_x\rangle$:

$$|\Phi, A_x\rangle_{\text{ph}} = \int D[\tilde{\Lambda}(x) U_{\tilde{\Lambda}}]\, |\Phi, A_x\rangle, \tag{41}$$

that is the set of all field configurations $\{\Phi, A_x\}$ splits into equivalence classes. Within each class the elements are related to one another by the small gauge transformations $\tilde{\Lambda}(x)$. In other words, due to the limitations imposed by the Gauss law the coordinates of the system are only the classes themselves, and for each class there is a physical field state $|\Phi, A_x\rangle_{\text{ph}}$. As is seen from (38), to each function $\exp\{i\alpha(x)\}$ there corresponds its own potential A_x. Imposing the boundary conditions on $\alpha(x)$ and compactifying the region of variation of the argument $-\infty < x < \infty$ by means of the condition $\exp\{i\alpha(\infty)\} = \exp\{i\alpha(-\infty)\}$ (making it equivalent to a circle), we see that all the functions $\exp\{i\alpha(x)\}$ map the circle into the group $U(1)$. It is known [11] that these mappings are split into homotopic sectors, which, as above, can be characterized by the "number of revolutions" N according to the expression

$$N = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\alpha}{dx} dx = \frac{1}{2\pi}[\alpha(\infty) - \alpha(-\infty)]. \tag{42}$$

If the above classification is carried out after the identification of the global gauge group, that is, for instance, $\alpha(-\infty) = 0$ is fixed for all configurations, then obviously $\alpha(\infty) = 2N\pi$, where N is the homotopic index (42).

The perturbative vacuum state is constructed around each homotopic class N of classical vacuum states as a wave functional $|N\rangle$ with a minimum energy. The states $|N\rangle$ with different N are called topological vacuum states; a topological vacuum state can be constructed in each homotopic sector N with the same energy. The topological vacuum states are degenerate. It is remarkable that there is tunneling between them so that none of the states $|N\rangle$ is a true vacuum state. A genuine vacuum state in the principal order can be constructed in the form of a linear superposition of these perturbative topological vacuum states, and this results in a nondegenerate state with respect to energy. These states are called θ-vacuum states.

The tunneling between the classical vacuum states that belong to different homotopic sectors is determined by the Euclidean functional integral

$$G(\tau) = \int D[\Phi(x, \tau')]D[\Phi^*(x, \tau')]D[A_\mu(x, \tau')] \exp\{-S_E\}, \tag{43}$$

where the Euclidean action is

$$S_E = \int_{-\infty}^{\infty} dx_1 \int_{-\tau/2}^{\tau/2} dx_2 \left\{ \frac{1}{4}F_{\mu\gamma}F^{\mu\gamma} + \frac{1}{2}|D_\mu\Phi|^2 + \frac{1}{4}\lambda'(|\Phi|^2 - F^2) \right\}. \tag{44}$$

Here $x_1 \equiv x$, $x_2 \equiv \tau'$, τ is the Euclidean "time", and μ and γ take on the values 1 and 2, and, when calculating the amplitude of the transition that connects two vacuum states (43), it is assumed that $\tau \to \infty$.

The boundary conditions on the fields are now imposed at the space-time perimeter, which is topologically equivalent to a circle and can be parametrized by means of an angle θ:

$$\Phi(x, \tau') \to F \exp\{i\alpha(\theta)\}, \quad A_\mu \to \frac{1}{ie} \exp\{-i\alpha(\theta)\}\partial_\mu \exp\{i\alpha(\theta)\}. \tag{45}$$

As above, each set of boundary conditions on the circle determines functions of the form of $\exp\{i\alpha(\theta)\}$, which can in turn be split into homotopic classes indexed by the number Q which has the meaning of the "number of revolutions":

$$Q = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\alpha}{d\theta} d\theta = \frac{e}{2\pi} \oint A\, dr = \frac{e}{4\pi} \int F_{\mu\gamma}\epsilon_{\mu\gamma}\, d^2x. \tag{46}$$

The numbers Q are called homotopic indices.

As a result of tunneling, the true vacuum states are defined as

$$|\theta\rangle = \sum_{N=-\infty}^{\infty} \exp\{iN\theta\}|N\rangle, \tag{47}$$

where θ is a parameter, each state $|\theta\rangle$ being a vacuum state belonging to a separate state sector. The states belonging to different sectors cannot be related to one another by means of gauge-invariant operators. The θ-vacuum energy is determined by the continual integral

$$\lim_{\tau\to\infty} \exp\{-E_\theta\tau\} = \lim_{\tau\to\infty} \int D[A_\mu, \Phi, \Phi^*]_{\text{all } Q} \exp\{-(S_E + S_f + i\theta Q)\}. \tag{48}$$

Here $D[A_\mu, \Phi, \Phi^*]_{\text{all } Q}$ means that the integration extends over the fields belonging to all sectors Q; and S_f is a term fixing the gauging. One can see that in formula (48) use is made of a modified θ-dependent action defined as

$$S_\theta = S_E + S_f + \frac{ie\theta}{4\pi} \int d^2x\, \varepsilon_{\mu\gamma} F_{\mu\gamma}. \tag{49}$$

This modification could have been constructed earlier by adding to the Lagrangian density (24) in the Minkowski space an additional θ-dependent term

$$\Delta\mathcal{L}_\theta = \frac{e\theta}{4\pi} \varepsilon_{\mu\nu} F^{\mu\nu}, \tag{50}$$

which would have made possible a different approach to the problem of the θ vacuum by considering θ as just a Lagrangian parameter. Although (50) can be represented as a divergence term:

$$\Delta\mathcal{L}_\theta = \theta\partial^\mu \left(\frac{e}{2\pi} \varepsilon_{\mu\nu} F^{\mu\nu} \right), \tag{51}$$

not affecting the classical equations of motion, it perturbs the C- and P-symmetries of the original Lagrangian in all sectors $\theta \neq 0$.

In the 1+1 dimensions the expression $(1/2)\varepsilon_{\mu\nu}F^{\mu\nu}$ is simply an electric field E_x which, upon reflection, is transformed as $E_x \to -E_x$. In our opinion, the break of the Lagrangian C-symmetry is related to the vacuum reconstruction resulting in the formation of Laughlin quasi-particles with fractional electric charges. Hence, if the parameter θ is introduced into the theory phenomenologically, the symmetry is broken dynamically.

The mean value of the Euclidean electric field in the θ vacuum is [11]

$$\langle F_{12}(x, \tau') \rangle_\theta = \frac{1}{2L\tau} \Big\langle \int \varepsilon_{\mu\gamma} F_{\mu\gamma} d^2x \Big\rangle_\theta, \tag{52}$$

where $L\tau \to \infty$ is the volume of the Euclidean space-time.

In the Minkowski space, instead of $\langle F_{12} \rangle_\theta$ we have

$$F_{01} \equiv E_x = \frac{4\pi}{e} \text{const} \cdot \sin\theta,$$

whence it is seen that the θ vacuum possesses a constant "background" electric field with a nonzero strength for $\theta \neq 0$ or π.

The concept of topological vacuum states $|N\rangle$ makes it possible to interpret these effects within the framework of the SSB theory. To this end we consider large gauge transformations with function $\Lambda_1(x)$ satisfying the boundary condition $\Lambda_1(\infty) = -2\pi \neq 0$. This transformation results in the following changes:

$$\begin{aligned} \alpha(\pm\infty) &\to \alpha(\pm\infty) + \Lambda_1(\pm\infty), \\ N &\to \frac{1}{2\pi}[\alpha(\infty) - \alpha(-\infty)] \to N - 1. \end{aligned} \tag{53}$$

The quantum version of these transformations is

$$T_1|N\rangle = |N-1\rangle, \tag{54}$$

where T_1 is the operator performing the gauge transformation with the function $\Lambda_1(x)$, under which the Hamiltonian is left invariant:

$$[T_1, \mathcal{H}] = 0. \tag{55}$$

As is seen from (54) and (55), the gauge symmetry of the theory using topological vacuum states is spontaneously broken by large gauge transformations. However, according to the general QFT theorems, this break of symmetry must lead to classical effects, namely, in the case under consideration, to the appearance of a macroscopic Hall current J_y. The physical interpretation then becomes quite clear: the inclusion of a weak electric field E_x in a real electron system is equivalent to the generation of large gauge transformations in the effective system with topological vacuum states $|N\rangle$. Tunneling gives rise to the θ vacuum, which is nondegenerate and has a "background" field.

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