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SPACE-TIME TOPOLOGY AS A CONSEQUENCE OF THE DYNAMICS OF CLOSED BOSON STRINGS. I. THE ZEROTH AND FIRST APPROXIMATIONS

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The possibility of constructing a theory in which the space-time topology is determined by the dynamics of closed boson strings is analyzed. The initial object in the theory is a topological space of the contours which determines the form of the string interaction vertex. A cohomological complex and a topologically invariant action are constructed on the space of contours. The equations of motion obtained by varying the action coincide with the condition of closure of differential cohomology forms. The analysis is carried out in the form of an expansion with respect to the string coupling constant. The zeroth- and first-order approximations are considered. Restrictions imposed on the space-time topology by the string dynamics are found. In what follows, by the space-time is meant an intermediate finite-dimensional space obtained from the original infinite-dimensional space of the contours and containing singularities. To pass to a smooth physical space-time it is necessary to pass to the massless limit, which will be the subject of another paper.

In the field theory the space-time topological structure is usually a background for interaction, and the dynamics of physical fields does not affect the topology. Closed strings make it possible to construct a theory in which the topology is a consequence of the string field dynamics. The present paper is devoted to one of the various approaches to the realization of this program. In this case use is made of an expansion with respect to the string coupling constant. We shall confine ourselves to taking account of the zeroth and first terms of the series that correspond to free strings and the tree approximation. The main stages of the construction are as follows. 1. A specific space of contours is introduced. 2. A local parametrization is performed, which corresponds to the zeroth and first orders of interaction, and the corresponding expression is found for the string interaction vertex in the given parametrization. 3. A tangent space is introduced and a cohomological complex is constructed, which consists of a nilpotent differential and closed differential forms. 4. Using the nilpotency of the differential, an algebra of differential forms on the space of contours and a geometrical equation are found, the latter being a closure condition for the differential forms. 5. A topologically invariant action of the theory is constructed for which the equation of motion coincides with the geometrical equation. 6. Using the condition of nilpotency of the differential, the restrictions on the admissible space-time topology are found.

We now proceed to a more detailed presentation.

1. To describe an infinite-dimensional manifold of contours M we define a set of coordinate neighborhoods and transition functions from one neighborhood to another. In this case there exists a covering of the contour space by open neighborhoods $\{U_\alpha\}$ such that on each neighborhood U_α a mapping Ψ_α is specified, which transforms each element $z \in U_\alpha$ into a set of continuous closed contours on an open neighborhood $u_\alpha \subset R^N$. The differentiable two-dimensional surfaces on u_α are mapped by Ψ_α^{-1} on smooth curves $z(\tau)$ in U_α. The transition function $\Psi_{\alpha\beta} = \Psi_\alpha \circ \Psi_\beta^{-1}$ preserves the differentiability of two-dimensional surfaces but may change the genus of a surface. Moreover, we require that for any surface of genus $g \neq 0$ on u_α there exist a transition function $\Psi_{\alpha\beta}$ such that it would transform the surface into a surface of genus zero on a neighborhood u_β. The transition functions thus describe simultaneously the manifold geometry and the contour dynamics. As is seen, the idea of this approach is based on the fact that a string is not an external object on the space-time but is a set of points belonging to the space-time and forming a closed contour.

Hence, the space-time is not a background for the string motion but results from the string dynamics.

2. To introduce differentiation on the contour space it is necessary to set a smooth parametrization of the neighborhoods U_α. (By a smooth parametrization is meant the one not violating the differentiability of two-dimensional surfaces.) To this end we choose the origin z on U_α and set a family of nonintersecting smooth curves $z(\tau)$, $z(\tau_0) = z$ covering the entire neighborhood U_α. The function Ψ_α maps smooth curves onto smooth surfaces on $u_\alpha \subset R^N$. To parametrize the contour space in the zeroth-order approximation we take into account that a free string in motion sweeps a cylinder, so we parametrize all cylindrical surfaces $z(\tau, \sigma)$: $\Psi_\alpha^{-1}(z(\tau, \sigma)) = z^0(\tau)$, $\tau_0 < \tau < \tau_1$. The family of all cylindrical surfaces on u_α does not cover the entire neighborhood U_α. We denote by U_α^0 the union of all curves $z^0(\tau)$; $U_\alpha^0 \subset U_\alpha$.

In the first order we parametrize all surfaces with branching: $z \to x,\ y$. To rule out the intersection of the curves corresponding to the first-order approximation with those belonging to U_α^0 it is necessary to introduce a deleted region on the neighborhood u_α. The indices of the mappings of the contours on the deleted region change upon interaction.

Proceeding from the requirement that the parametrization of the interaction must not violate the surface differentiability, we arrive at the following expression for the interaction vertex:

$$
\begin{aligned}
V(x,\, y,\, z) &= \delta(n_x + n_y + n_z)\theta(\sigma_z^b - \sigma_z^a) \\
&\times \Bigg\{ \prod_{\sigma_z^a < \sigma < (\sigma_z^a + 2\pi\alpha_3)} \delta(z(\sigma) - x(4\pi\alpha_3 - \sigma)) \prod_{(\sigma_z^a + 2\pi\alpha_3) < \sigma < (\sigma_z^a + 2\pi\overline{\alpha}_3)} \delta(z(\sigma) \\
&- y(2\pi\overline{\alpha}_2 - \sigma)) \prod_{(2\pi\alpha_2 - \sigma_z^a) < \sigma < (2\pi(\alpha_2 + \alpha_3) - \sigma_z^a)} \delta(y(\sigma) - x(2\pi(3\alpha_3 + \alpha_2) - 2\sigma_z^a \\
&- \sigma)) \Bigg\} - \delta(n_x + n_y + n_z)\theta(\sigma_z^a - \sigma_z^b)\{\sigma_z^a \leftrightarrow \sigma_z^b\},
\end{aligned}
\tag{1}
$$

where $n_{x,y,z}$ are the indices of the contour mappings; the parametrization lengths α satisfy the relations

$$\alpha_1 + \alpha_3 = \overline{\alpha}_1; \quad \alpha_1 + \alpha_2 = \overline{\alpha}_2; \quad \alpha_2 + \alpha_3 = \overline{\alpha}_3.$$

For the sake of simplicity we assume, without loss of generality, that

$$\alpha_3 = \theta(\sigma_z^b - \sigma_z^a)\alpha_1 + \theta(\sigma_z^a - \sigma_z^b)\alpha_2.$$

The function $\delta(n_x + n_y + n_z)$ describes the "conservation law" for the mapping indices and thus characterizes the topological properties of the contour manifold. The function θ takes into account the possibility of different choice of the point of reference for the parameter σ on the string z. The other δ functions describe the multilocal interaction. The requirement of differentiability determines the range of variation of the parameters σ.

3. Any functional $f(z)$ on U_α^0 must not depend on the above parametrization, i. e., the condition

$$L_\sigma f(z(\sigma')) = 0 \tag{2}$$

holds, where L_σ are the reparametrization generators with respect to the parameter σ.

We now introduce a tangent space on U_α^0. We define the tangent vector to the surface $z(\sigma, \tau)$ as a derivative with respect to the parameter:

$$\partial f \equiv \lim_{\tau \to \tau_0} \frac{f(z(\sigma,\, \tau)) - f(z(\sigma,\, \tau_0))}{\tau - \tau_0}.$$

Since f satisfies condition (2), for any parametrization $\sigma \to \sigma'(\sigma, \tau)$ $\tau' = \tau$ we have

$$\partial' f \equiv \lim_{\tau \to \tau_0} \frac{f(z(\sigma',\, \tau)) - f(z(\sigma',\, \tau_0))}{\tau - \tau_0} = \partial f.$$

Consequently, the differential operator on the contour space contains the whole collection of operators of the type of ∂ with respect to all reparametrizations. Therefore the general expression for a differential form on the neighborhood U_α^0 is

$$\omega = \omega(z) + \omega^{\sigma^{(+)}}(z)c(\sigma^{(+)}) + \omega^{\sigma_1^{(+)}\sigma_2^{(+)}}(z)c(\sigma_1^{(+)})c(\sigma_2^{(+)}) + \omega^{\sigma^{(-)}}(z)c(\sigma^{(-)}) + \omega^{\sigma^{(+)}\sigma^{(-)}}(z)c(\sigma^{(+)})c(\sigma^{(-)}) + \dots + \omega^{\sigma_1^{(-)}\sigma_2^{(-)}}(z)c(\sigma_1^{(-)})c(\sigma_2^{(-)}) + \dots \equiv \omega(z, c),$$

where $\sigma^{(\pm)} = \sigma \pm \tau$ and $c(\sigma)$ are anticommuting quantities satisfying the relation

$$\left\{ c(\sigma_1^{(\pm)}),\, c(\sigma_2^{(\pm)}) \right\}_{(+)} = \sum_{\sigma_3^{(\pm)}} K^{\sigma_3^{(\pm)}}_{\sigma_1^{(\pm)}\sigma_2^{(\pm)}} c(\sigma_3^{(\pm)}).$$

The condition of reparametrization invariance for the differential forms ω is written

$$(d^0\omega)(z, c) = 0, \tag{3}$$

where

$$d^0 = c(\sigma^{(+)})L(\sigma^{(+)}) + c(\sigma^{(-)})L(\sigma^{(-)}) + K^{\sigma_3^{(\pm)}}_{\sigma_1^{(\pm)}\sigma_2^{(\pm)}} c(\sigma_1^{(\pm)})c(\sigma_2^{(\pm)})\bar{c}(\sigma_3^{(\pm)}).$$

Relation (2) contains the reparametrizations with respect to both σ and τ and implies the closure of the form ω for noninteracting strings (in the zeroth-order approximation).

The general expression for a form on the neighborhood U_α^1 (in the first-order approximation) is determined by the three variables x, y, and z and is written as

$$\omega = \{\omega(z, c_z);\ \omega(x, c_x),\ \omega(y, c_y)\},$$

and the closure condition (analogous to (3)) takes the form

$$(d^1\omega)(z, c_z) = 0, \tag{3'}$$

where

$$(d^1\omega)(z) = (d^0\omega)(z) + F^1_{(z)}(\omega(x)) + F^1_{(z)}(\omega(y)).$$

In formula (3′) the contours x and y together with z form a triplet, and F^1 are linear functionals on the set of differential forms that are determined by the properties of the contour space. From the linearity of F^1 it follows that the forms can be represented as a linear combination

$$F^1_{(z)}(\omega(x), \omega(y)) = g(\Phi * \omega)(z) \equiv \sum_{x,y} g\Phi(y)\omega(x)V(x, y, z),$$

where Φ is normalized to unity and g is a linear factor which is identified in what follows with the string coupling constant. The differential form Φ is determined by the properties of the contour space and will therefore be interpreted as an expression describing the string field.

4. The property of first-order nilpotency of the differential

$$[d^1]^2 = 0$$

implies the following relations:

$$[d^0]^2 = 0, \tag{4a}$$
$$d^0(\Phi * \omega) = (d^0\Phi) * \omega - \Phi * d^0\omega, \tag{4b}$$
$$\omega * \Phi = \Phi * \omega, \tag{4c}$$
$$(\Phi * \Phi) * \omega - \Phi * (\Phi * \omega) = 0, \tag{4d}$$
$$d^1\Phi = d^0\Phi + g\Phi * \Phi = 0. \tag{4e}$$

Formulas (4b)-(4d) express the rules of operations on the functionals. Formula (4e) means the closure of the form Φ, i. e., Φ does in fact describe the geometrical properties of the contour space and does not depend on the parametrization.

To find the transformation law for the form Φ upon transition from one neighborhood to another we use the invariance property of the differential with respect to these transformations. It can be shown that the form is transformed in the following way:

$$(d^1\Phi)(z) = (d^1\Psi)(z) = d^0\Psi + g\Phi * \Psi, \tag{5}$$

where $\Psi(z, c)$ is an infinitesimal transition function. This means that the theory is invariant with respect to the addition of an exact form to a closed one, i. e., the set of all Φ constitutes the cohomology group of the contour space.

5. To obtain a topological invariant, i. e., an expression invariant relative to transformation (5), it is necessary to introduce the notion of integral on the manifold M. Since we are dealing with a linearized theory (in the first order), as a region of integration we must take the tangent space in the first order with respect to the coupling constant. In the integral there appear two terms corresponding to the measures on U^0_α and U^1_α/U^0_α:

$$\int\limits_{U^1_\alpha} Dz = \int\limits_{\tau_0}^{\tau_1} d\tau \oint Dz(\sigma, \tau) + \int\limits_{\tau_0}^{\tau_1} d\tau \int\limits_{\tau}^{\tau_1} d\tau' d\tau'' \oint Dz(\sigma, \tau) v(x, y, z)$$
$$\times \oint Dx(\sigma, \tau') \oint Dy(\sigma, \tau''),$$

where

$$v(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \text{ form a triplet,} \\ 0 & \text{in the other cases.} \end{cases}$$

As can be shown by direct verification, an expression invariant with respect to (5) is represented as

$$\Gamma^1 = \int\limits_{U^0} Dz \, \langle \Phi, d^0\Phi \rangle(z) + g \int\limits_{U^0} Dz \, \langle \Phi, \Phi * \Phi \rangle(z)$$
$$= \int\limits_{U^1} \langle \Phi, d^0\Phi \rangle + \frac{2}{3} g \int\limits_{U^1} \langle \Phi, \Phi * \Phi \rangle,$$

where $\langle \Phi, \tilde{\Phi} \rangle(z) = \Phi(z, c_z)\tilde{\Phi}(z, \tilde{c}_z) \prod\limits_{0<\sigma<2\pi} \delta(c(z) - \tilde{c}(2\pi - z))$ is the scalar product.

For the variation of the invariant Γ^1 we find

$$\delta\Gamma^1 = (d^1\Phi)\delta\Phi. \tag{7}$$

Since Φ is a closed form, expression (7) vanishes. Consequently, the values of Γ^1 form no continuum, and the invariant assumes a number of discrete values.

From the physical standpoint, the topological invariant Γ^1 is action. It is important to stress that the equation of motion (7) coincides with the closure condition for the form Φ, and hence the contour dynamics is determined by the cohomologies of the space M. The fact that the dynamic equation (7) coincides with the geometrical equation (4e) indicates that the scheme is self-consistent.

Note that Γ^1 is a tree approximation to the effective action. Similarly, the second order corresponds to the one-loop effective action, etc. Thus we have at once obtained a second-quantized theory. Then, if the total differential d is found, the corresponding total invariant Γ determines the non-perturbative formulation of the theory.

The suggested approach allows one to consider the problem of vacuum state from a new viewpoint. By definition, by vacuum is meant a state containing no field interaction. In our approach this corresponds to the zeroth-order approximation and, consequently, to a "flat" space-time. The appearance of interaction is accompanied by a change in the space-time topology, and thus the ordinary notion of vacuum state loses its meaning.

6. We now consider the topological properties of the space-time. The transition functions between the neighborhoods u_α in the space R^N may be assumed as corresponding to the transition functions $\Psi_{\alpha\beta}$ on the contour space. The neighborhoods $\{u_\alpha\}$ form an atlas of a manifold that we identify with the space-time. In the zeroth order describing the noninteracting strings the neighborhood u_α is homeomorphic to a disk, and consequently to this approximation there corresponds a "flat" space-time.

For the interaction to exist it is necessary that the neighborhood u_α contain holes, therefore in the first-order approximation the topology must be nontrivial. The only restriction imposed on the admissible space-time in the first-order approximation is that there exist contours that cannot be contracted to a point, i. e., the condition for nontriviality of the first homotopic group π_1. As will be shown in our further publications, taking account of higher approximations enables one to specify more precisely the admissible space-time topology.

Thus, we have demonstrated the possibility of constructing an approach in which the space-time topology is determined by the contour dynamics. The suggested approach has a number of specific features: (i) it leads to certain restrictions for the admissible space-time topology; (ii) it gives at once a second-quantized theory and an effective action. The action is a topological invariant, thus allowing the study of non-perturbative effects, and describes only the contour self-action; (iii) the problem of vacuum state acquires a specific form; (iv) the nilpotency of the differential in all orders with respect to the coupling constant implies the absence of anomalies.

The geometrical approach to the field theory of strings is employed in a number of works [1-5]. In all of them the contour dynamics is described by means of fibering theory, and the interaction takes place in the fiber. In the suggested approach the fibering theory is not used and the interaction takes place in the contour space itself. It is this property that makes it possible to obtain the topology as a consequence of the dynamics of closed strings.

Among the most interesting unsolved problems we should first of all mention the inclusion of the contributions from higher approximations with respect to the coupling constant. This seems to elucidate some questions relating to the compactification and the passage to the physical space-time. Of interest also is the investigation of some general properties such as renormalizability, unitarity, etc. Finally, it is necessary to generalize the proposed approach to superstrings and heterotic strings. It is likely that in these cases, too, some additional restrictions for possible space-time topologies will appear.

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