

SPACE-TIME TOPOLOGY AS A CONSEQUENCE OF THE DYNAMICS OF CLOSED SUPERSTRINGS

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The results obtained in the previous papers for a closed boson string are generalized to the case of a closed superstring.

In the previous papers [1, 2] we described an approach in which the space-time topology is determined by the dynamics of closed strings, and also considered closed boson strings. The purpose of the present paper is to apply the suggested approach for analyzing closed superstrings in the second-order approximation with respect to the coupling constant. In accordance with the method, de Rahm's cohomological complex will be constructed on the contour superspace, a topological invariant will be found in the given approximation, and also the constraints on the space-time topology will be obtained.

1. To describe an infinite-dimensional supermanifold G of contours we set a boson space M of contours and a sheaf of supercommutative rings Θ on M (see, e.g., [3]). This means that on each neighborhood U_α introduced in [1] the sheaf $\Theta_{U_\alpha}^0 \otimes \Theta_{U_\alpha}^1$ is defined, where Θ^0 and Θ^1 are the sheafs of even and odd functionals on U_α . Besides, for each neighborhood U_α there exists a mapping Ψ_α such that

$$\Psi_\alpha(\Theta^0 \otimes \Theta^1)_z = \Theta_{z_i(\sigma)}^0 \otimes \Theta_{z_j(\sigma)}^1, \quad (1)$$

where $z \in U_\alpha$ and $z_i(\sigma)$ are the closed contours in the space R^N .

Under the mapping Ψ_α the smooth curves on the supermanifold are mapped into a differentiable two-dimensional surface on $u_\alpha \subset R^N$, and on each of them a supercommutative sheaf of functionals is specified. The transition function $\Psi_{\alpha\beta} = \Psi_\beta \circ \Psi_\alpha^{-1}$ preserves the differentiability of two-dimensional surfaces but may change the genus of a surface. For any surface of genus $g \neq 0$ on u_β there exists a transition function $\Psi_{\beta\alpha}$ such that it would transform the surface into a surface of genus zero on u_α . Hence, the transition functions describe simultaneously the supermanifold G geometry and the contour dynamics.

2. To define the differentiation on the contour superspace it is necessary to introduce a smooth parametrization of the neighborhoods U_α^s . To this end we choose the origin z^s on U_α^s and set a family of nonintersecting smooth curves $z(\tau)$, $\tau_0 < \tau < \tau_1$ covering the entire neighborhood U_α . In the family we separate out all curves $z^0(\tau)$ that correspond to smooth surfaces of genus zero without branchings and cover the entire neighborhood $U_\alpha^0 \subset U_\alpha$. The surfaces $\Theta_{z^0(\sigma,\tau)}^0 \otimes \Theta_{z^1(\sigma,\tau)}^1$ extend the parametrization to all contours lying in U_α^0 . This parametrization corresponds to the linearized theory in the zeroth order with respect to the interaction and describes the tangent space to the space of contours with the zeroth order of contact.

In the first order we parametrize all surfaces with branching $z^s \rightarrow x^s, y^s$. In terms of the sheafs the branching is written in the following way:

$$\begin{aligned} \Theta_z &\rightarrow \Theta_x, \Theta_y, \\ \Theta_z^0 &= V^0(\Theta_x \Theta_y \Theta_z) \Theta_x^0 \Theta_y^0 \otimes V^1(\Theta_x \Theta_y \Theta_z) \Theta_x^1 \Theta_y^1, \\ \Theta_z^1 &= V^1(\Theta_x \Theta_y \Theta_z) (\Theta_x^0 \Theta_y^1 \otimes \Theta_x^1 \Theta_y^0). \end{aligned} \quad (2)$$

On condition that the space is locally splittable, the vertex can be represented in the form

$$\begin{aligned} V^0 &= V(x, y, z) V(\vartheta, \xi, \lambda), \\ V^1 &= P(\sigma^a, \sigma^b) V(x, y, z) \tilde{V}(\vartheta, \xi, \lambda). \end{aligned}$$

Here the expression for $V(x, y, z)$ given in [1], formula (1), is used. The limits of variation of σ in the expressions for $V(x, y, z)$ and $V(\vartheta, \xi, \lambda)$ coincide, and the operator $P(\sigma^a, \sigma^b)$ transforms the elements from

Θ^0 into Θ^1 and vice versa. The boundary conditions for the contours ϑ , ξ , and λ having a nonzero mapping index can be of two kinds, namely $\xi(0) = \pm\xi(2\pi)$. Therefore, at a definite point σ , in the expression for the vertex $\tilde{V}(\vartheta, \xi, \lambda)$ the signs of ϑ , ξ , and λ are pairwise changed, i. e.,

$$\tilde{V}(\vartheta, \xi, \lambda) = V(\vartheta, \tilde{\xi}, \tilde{\lambda}) + V(\tilde{\xi}, \vartheta, \tilde{\lambda}) + V(\tilde{\xi}, \tilde{\vartheta}, \lambda).$$

3. Any element of the ring Θ_z must not depend on the introduced parametrization, i. e., the condition

$$(L_\sigma + G_\sigma) \begin{pmatrix} f(z(\sigma)) \\ h(z(\sigma)) \end{pmatrix} = \begin{pmatrix} (L'_\sigma + G'_\sigma)f(z(\sigma)) \\ (L''_\sigma + G''_\sigma)h(z(\sigma)) \end{pmatrix} = 0 \quad (3)$$

must hold, where L'_σ and L''_σ are the generators of reparametrizations along σ on Θ_z^0 and Θ_z^1 for the $z(\sigma)$ -coordinates, and G'_σ and G''_σ are the same for the $\xi(\sigma)$ -coordinates.

We now introduce a tangent space on U_α^0 . By analogy with [1], we define the tangent vector to the surface $z(\sigma, \tau)$, $\vartheta(\sigma, \tau)$ as a derivative with respect to the parameter:

$$\begin{pmatrix} \partial f \\ \partial h \end{pmatrix} = \begin{cases} \lim_{\tau \rightarrow \tau_0} \frac{f(z(\sigma, \tau), \vartheta(\sigma, \tau)) - f(z(\sigma, \tau_0), \vartheta(\sigma, \tau_0))}{\tau - \tau_0}, \\ \lim_{\tau \rightarrow \tau_0} \frac{h(z(\sigma, \tau), \vartheta(\sigma, \tau)) - h(z(\sigma, \tau_0), \vartheta(\sigma, \tau_0))}{\tau - \tau_0}. \end{cases}$$

Taking into account that the tangent space should be invariant with respect to the parametrization, we write down the general expression for a differential form on the neighborhood U_α^0 :

$$\Omega^0 = \begin{pmatrix} \omega \\ \tilde{\omega} \end{pmatrix} = \begin{pmatrix} \omega(z, c, \vartheta, \beta) \\ \tilde{\omega}(z, c, \vartheta, \beta) \end{pmatrix},$$

where β are commuting quantities describing the tangent differentials in the ϑ -direction. The condition of reparametrization invariance for differential forms is written in the following way:

$$d^0 \Omega^0 = 0,$$

where

$$d^0 = L_\sigma^{(\pm)} c^{(\pm)}(\sigma) + G_\sigma^{(\pm)} \beta^{(\pm)}(\sigma) + K + \tilde{K}. \quad (4)$$

Up to now there was no need in distinguishing between RNS- and GS-strings. The choice of the differential in the form of (4) means that we confine ourselves to considering an RNS-string.

The general expression for a differential form on the neighborhood U_z^1 (i. e., in the first-order approximation) is determined by three pairs of variables: $z, \vartheta; x, \xi; y, \lambda$ and is written as

$$\Omega^1 = \{\Omega^0(z, \vartheta) + \Omega^0(x, \xi) + \Omega^0(y, \lambda)\},$$

and, by analogy with (4), the closure condition takes the form

$$d^1 \Omega^1 = d^0 \Omega^0(z, \vartheta) + F_1(\Omega^0(x, \lambda), \Omega^0(y, \xi)) = 0, \quad (5)$$

where

$$F_1(\Omega^0) = F_1 \begin{pmatrix} \omega \\ \tilde{\omega} \end{pmatrix} = \begin{pmatrix} F_1(\omega(x, \xi)) + \tilde{F}_1(\tilde{\omega}(x, \xi)) \\ \tilde{F}_1(\tilde{\omega}(y, \lambda)) + \tilde{F}_1(\omega(y, \lambda)) \end{pmatrix}. \quad (6)$$

are linear functionals on the set of tangent forms with range in the set T_{U^1/U^0}^* of tangent forms $T_{U^0}^*$.

Taking into account decomposition (2) of sheafs Θ upon the interaction of contours and the linearity of functionals F we rewrite (6) in the form

$$F(\Omega^0) = g\Phi * \Omega^0 = g \begin{pmatrix} \varphi * \omega + \tilde{\varphi} * \tilde{\omega} \\ \varphi * \tilde{\omega} + \tilde{\varphi} * \omega \end{pmatrix}.$$

The differential form $\Phi = \begin{pmatrix} \varphi \\ \tilde{\varphi} \end{pmatrix}$ is completely determined by the topological structure of the space M , and we shall interpret it as a quantity describing the string field.

4. The total differential on the neighborhood U_α^s of the contour supermanifold has the form

$$\begin{aligned} d\Omega &= d^0\Omega^0 + F_1(\Omega^0) + F_2(\Omega^0) + \dots \\ &= d^0 \begin{pmatrix} \omega \\ \tilde{\omega} \end{pmatrix} = \begin{pmatrix} F_1(\omega) + F_2(\omega) + \dots + \tilde{F}_1(\tilde{\omega}) + \tilde{F}_2(\tilde{\omega}) + \dots \\ \tilde{F}_1(\tilde{\omega}) + \tilde{F}_2(\tilde{\omega}) + \dots + \tilde{F}_1(\omega) + \tilde{F}_2(\omega) + \dots \end{pmatrix}. \end{aligned} \quad (7)$$

This form of the differential is due to the fact that at the very beginning the theory is linearized and the successive approximations with respect to the coupling constant are investigated, every successive approximation corresponding to the approximation of the neighborhood U_α^s by a tangent space with a definite order of contact.

As in [1], we require that the differential should be nilpotent on condition that the form is closed:

$$\begin{aligned} [d]^2\Omega &= g(d\Phi) * \Omega^0, \\ [d]^2\Omega &= [d^0]^2\Omega^0 + [d^0 F_1(\Omega^0) + F_1(d^0\Omega^0) + F_1(F_1(\Omega^0))] \\ &\quad + [d^0 F_2(\Omega^0) + F_2(d^0\Omega^0) + F_1(F_2(\Omega^0)) + F_2(F_1(\Omega^0))] + \dots \end{aligned}$$

Collecting terms in like powers of the constant g we obtain a system of equations for the functionals F_i and the differential d^0 :

$$\begin{cases} [d^0]^2 = 0, \\ d^0(\Phi * \Omega) + \Phi(d^0\Omega) + g\Phi * (\Phi * \Omega) = (d^0\Phi + g\Phi * \Phi) * \Omega, \\ d^0 F_2(\Omega) + F_2(d^0\Omega) + \Phi * F_2(\Omega) + F_2(\Phi * \Omega) = F_2(\Phi) * \Omega. \\ \dots \end{cases} \quad (8)$$

The first equation in system (8) is the nilpotency condition for the zeroth-order differential.

The second equation in (8) implies the rules of operations on functionals:

$$\begin{aligned} d^0(\Phi * \Omega) &= (d^0\Phi) * \Omega - \Phi * d^0\Omega, \\ \Phi * (\Phi * \Omega) &= (\Phi * \Phi) * \Omega, \\ \Phi * \Omega + \Omega * \Phi &= 0. \end{aligned} \quad (9)$$

Whence follow the differentiation rules for even and odd components:

$$\begin{aligned} d^0(\varphi * \tilde{\omega}) &= (d^0\varphi) * \tilde{\omega} - \varphi * d^0\tilde{\omega}, \\ d^0(\tilde{\varphi} * \tilde{\omega}) &= (d^0\tilde{\varphi}) * \tilde{\omega} - \tilde{\varphi} * d^0\tilde{\omega}, \end{aligned}$$

and also the property of associativity for the composition of functionals:

$$\begin{aligned} (\varphi * \tilde{\varphi}) * \tilde{\omega} &= \varphi * (\tilde{\varphi} * \tilde{\omega}); & \tilde{\varphi} * (\tilde{\varphi} * \omega) &= (\tilde{\varphi} * \tilde{\varphi}) * \omega, \\ \tilde{\varphi} * (\varphi * \tilde{\omega}) &= (\tilde{\varphi} * \varphi) * \tilde{\omega}; & \varphi * (\varphi * \omega) &= (\varphi * \varphi) * \omega. \end{aligned}$$

5. To solve the third equation in system (8) we split it into two equations:

$$\begin{cases} d^0 F_2(\Omega) = -F_2(d^0\Omega), \\ \Phi * F_2(\Omega) = F_2(\Phi) * \Omega - F_2(\Phi * \Omega). \end{cases} \quad (10)$$

As in [1], we introduce the operator of dual differentiation

$$\bar{d}d\Omega' = \Omega', \quad d\bar{d}\Omega'' = \Omega'', \quad [\bar{d}]^2\Omega = [\bar{d}\Phi] \circ \Omega, \quad (11)$$

where

$$\bar{d}\Omega = \bar{d}^0\Omega^0 + \bar{F}_1(\Omega^0) + \bar{F}_2(\Omega^0) + \dots$$

Writing (11) in full we obtain a system of equations for \bar{F} analogous to (8) and a system of equations for F and \bar{F} supplementing (8) and (10). Since the algebra of functionals Φ and Ω coincides with the algebra of φ and ω in [1], for F_2 we obtain an expression similar to formula (10) in [2]:

$$\begin{aligned} F_2(\Omega) &= g^2 \{ \bar{d}^0 [\Phi * ((\bar{d}^0 \Phi) \circ d^0 \Omega)] - \Phi \circ \bar{d}^0 (\Phi * \Omega) \} \\ &= g^2 \left\{ \bar{d}^0 \left(\begin{aligned} &\varphi * ((\bar{d}^0 \varphi) \circ d^0 \omega + (\bar{d}^0 \tilde{\varphi}) \cdot d^0 \tilde{\omega}) + \tilde{\varphi} * ((\bar{d}^0 \varphi) \cdot d^0 \tilde{\omega} + d^0 \tilde{\varphi} \cdot d^0 \tilde{\omega}) \\ &\varphi * ((\bar{d}^0 \varphi) \cdot d^0 \tilde{\omega} + \bar{d}^0 \tilde{\varphi} \cdot d^0 \omega) + \tilde{\varphi} * ((\bar{d}^0 \varphi) \circ d^0 \omega + \bar{d}^0 \tilde{\varphi} \cdot d^0 \tilde{\omega}) \end{aligned} \right) \right. \\ &\quad \left. - \left(\begin{aligned} &\varphi \circ (\bar{d}^0(\varphi * \omega) + \bar{d}^0(\tilde{\varphi} * \omega)) + \tilde{\varphi} \cdot (\bar{d}^0(\varphi * \tilde{\omega}) + \bar{d}^0(\tilde{\varphi} * \tilde{\omega})) \\ &\varphi \cdot (\bar{d}^0(\varphi * \tilde{\omega}) + \bar{d}^0(\tilde{\varphi} * \tilde{\omega})) + \tilde{\varphi} \cdot (\bar{d}^0(\varphi * \omega) + \bar{d}^0(\tilde{\varphi} * \tilde{\omega})) \end{aligned} \right) \right\}. \end{aligned} \quad (12)$$

6. To find the transformation law for the form Φ upon an infinitesimal transition from one neighborhood to another we use the invariance property of the total differential with respect to these transformations. This results in

$$\delta \Phi = d\Psi + g\Phi * \Psi + F_2(\Psi) + \dots, \quad (13)$$

where $\Psi = \begin{pmatrix} \varphi \\ \tilde{\varphi} \end{pmatrix}$ is an infinitesimal transition function.

Equation (13) means that the theory is invariant with respect to the addition of an exact form $d\Psi$ to the closed form Φ , i. e., all solutions of the geometrical equations are generators of the first cohomology group of the contour space. In particular, if $\Psi \in T_{U_\alpha}^1$, then in the first order the transformation is written in the following way:

$$\delta_1^{(1)} \Phi = \begin{pmatrix} \delta_1^{(1)} \varphi \\ \delta_1^{(1)} \tilde{\varphi} \end{pmatrix} = \begin{pmatrix} g\tilde{\varphi} * \tilde{\psi} \\ d^0 \psi + g\varphi * \tilde{\varphi} \end{pmatrix}.$$

The scalar product of forms is written as

$$\langle \Phi, \Omega \rangle = \langle \varphi, \omega \rangle + \langle \tilde{\varphi}, \tilde{\omega} \rangle,$$

where $\langle \varphi, \omega \rangle$ is described in [1].

The action in the first order, which is invariant with respect to transformation (13), has the following form:

$$\begin{aligned} \Gamma_s^1 &= \int \langle \Phi, d^0 \Phi \rangle + \frac{2}{3} g \int \langle \Phi, \Phi * \Phi \rangle = \int \langle \varphi, d^0 \varphi \rangle + \int \langle \tilde{\varphi}, d^0 \tilde{\varphi} \rangle \\ &\quad + \frac{2}{3} g \left[\int \langle \varphi, \varphi * \varphi \rangle + \int \langle \varphi, \tilde{\varphi} * \tilde{\varphi} \rangle + \int \langle \tilde{\varphi}, \varphi * \tilde{\varphi} \rangle + \int \langle \tilde{\varphi}, \tilde{\varphi} * \varphi \rangle \right]. \end{aligned} \quad (14)$$

In conclusion we note one of the main properties of the above geometrical theory of superstrings. The thing is that the spectrum of massless states of a superstring in the algebraic approach is a supergravitation multiplet, and the gauge fields with spin 1 appear only in the theory of a heterotic string. In the approach presented here fields with spin 1 inevitably appear in the massless spectrum. Indeed, the theory of interacting strings assumes that in the parametric neighborhood $u_\alpha \subset R^N$ there exist "holes" and string states with nonzero torsion index n_i . Therefore $\alpha_\mu^{-1} |n_i\rangle$, $\sum_i n_i^2 = 1$ are massless states with spin 1. In this case using the string expansion with respect to the modes it is possible to solve the problem of local splittability, namely the functionals $f(z)$ on U_α^s are infinite-dimensional tensor fields $f(z_0, \alpha, \bar{\alpha}, \gamma, \bar{\gamma})$ on u_α that are factorized as a tensor product $f_R \otimes f_L = f(z_0, \alpha, \gamma) \otimes f(z_0, \bar{\alpha}, \bar{\gamma})$, where f_L is a locally splittable superfield and f_R has the mapping index n_i . Hence, there is no need to consider the theory of a heterotic string because in this approach the superstring is in fact heterotic.

REFERENCES

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