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ISOMETRIC IMMERSIONS OF ROTATION METRICS ON A SPHERE INTO E^4 IN THE FORM OF SURFACES OF REVOLUTION

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In this paper, under investigation is the problem of immersion of a metric on S^2, which has the form $ds^2 = g(r)(dr^2 + r^2 d\varphi^2)$ in a certain stereographic projection, into E^4 in the form of a surface of revolution with a pole. Denote the poles as O_1 and O_2. It is established that there exists an immersion in the class $C^1(S^2) \bigcap C^2(S^2 \setminus (O_1 \bigcup O_2))$. Some necessary and sufficient conditions are given for an immersion in the class $C^2(S^2)$. The deformability of the resulting surfaces in the same class is proved.

In the present paper we consider the possibility of isometric immersions of a metric of the form of

$$g(r)(dr^2 + r^2 d\varphi^2), \tag{1}$$

which is defined on a sphere and has a negative curvature at the poles, into E^4 in the form of a surface of revolution. In E^3 this is impossible.

Take a sphere in E^3 with stereographic projection on a plane. Let M be a manifold of class C^3 diffeomorphic to the sphere. Its coordinate atlas includes the charts of all stereographic projections of the sphere. For brevity, we shall speak of the stereographic projection of M with poles O_1 and O_2 and charts 1 and 2. To the points O_1 and O_2 there correspond points on the sphere, and the line (O_1, O_2) will be called the projection axis. Let g_{nm} be a Riemann metric of class $C^2(M)$. We shall call g_{nm} a rotation metric if there exists a stereographic projection of M such that in chart 1 in the Cartesian coordinates $\{x, y\}$ with origin at O_1 we have $g_{nm} = g(\sqrt{x^2 + y^2})\delta_{nm}$. It is easy to see that in chart 2 in Cartesian coordinates with origin at O_2 we have $g_{nm} = \tilde{g}(\sqrt{x^2 + y^2})\delta_{nm}$. We shall call $F : M \to E^4$ an immersion of M into E^4 in the form of a surface of revolution if there exists a stereographic projection such that if $\{u, t\}$ are polar coordinates in chart 1 with origin at O_1, then there are the Cartesian coordinates $\{x^1, x^2, x^3, x^4\}$ in E^4 such that F is determined by the formulas

$$\begin{aligned} x^1 &= r_1(u)\cos(\gamma_1 t + f_1(u)), \\ x^2 &= r_1(u)\sin(\gamma_1 t + f_1(u)), \\ x^3 &= r_2(u)\cos(\gamma_2 t + f_2(u)), \\ x^4 &= r_2(u)\sin(\gamma_2 t + f_2(u)), \end{aligned} \tag{2}$$

where $\gamma_1, \gamma_2 \in N$ and $r_1, r_2, f_1, f_2 \in C^2[0, +\infty)$.

Let g_{nm} be a rotation metric with negative curvature at O_1 and O_2. Let there exist $g^{(k)}(0)$ and $\tilde{g}^{(k)}(0)$, $k = \overline{1, 5}$. Then we have the following

Theorem.

1. *There exists an isometric immersion of the metric g_{nm} into E^4 in the class $C^1(M) \bigcap C^2(M \setminus (O_1 \bigcup O_2))$ in the form of a surface of revolution.*

2. *In the class $C^2(M)$ a necessary condition for the existence of an isometric immersion in the form of a surface of revolution is the requirement that $\forall u \in [0, +\infty)((\alpha(u) + 1)^2 \leqslant 4)$, and a sufficient condition is $\forall u \in [0, +\infty)((\alpha(u) + 1)^2 < 4)$, where $\alpha(u) = \dfrac{ug'(u)}{2g(u)}$.*

Proof. We shall prove an auxiliary assertion. Let us show that all projections, in which $g_{11} = g_{11}(\sqrt{x^2 + y^2})$ for the metric in question, have the same axis. It is obvious that $g(u) = \lambda(u)g_0(u)$, where

$g_0(\sqrt{x^2+y^2})\delta_{nm}$ is the metric of the sphere. Since $K(O_1) < 0$, we have $\lambda \not\equiv \text{const}$. We pass to a projection with another axis in which $g_{11} = g_{11}(\widetilde{u})$ (this is a displacement of g_0): $\widetilde{g}(\widetilde{u}, \widetilde{t}) = g_0(\widetilde{u})\lambda(u(\widetilde{u}, \widetilde{t}))$. However, we have $\partial u/\partial\widetilde{t} \neq 0$ for almost all $\widetilde{u}$ and almost all $\widetilde{t}$, which contradicts the condition $g_{11} = g_{11}(\widetilde{u})$. This proves the above assertion. Therefore, if the desired immersion F exists, then the projections in which the metric has the form of (1) and F has the form of (2) have a common axis. In this case the transition from one projection to another is described by the formulas $\widetilde{u} = \delta u$, $\widetilde{t} = t$ where $\delta \in R^1$. Here $\widetilde{\alpha}(\widetilde{u}) = \alpha(\widetilde{u}/\delta)$. Therefore, if $(\alpha+1)^2 \leqslant 4$, then $(\widetilde{\alpha}+1)^2 \leqslant 4$. Consequently, to prove the first part of assertion 2 of the theorem it is sufficient to show that if in some projection the metric has the form of (1) and the immersion has the form of (2), then in this projection $(\alpha+1)^2 \leqslant 4$. To prove assertion 1 and the second part of assertion 2 we need only take some Cartesian coordinates in E^4 and a projection with a metric of the form of (1) and to construct an immersion of the form of (2).

We set the following problem:

$$\{1\}\begin{cases} r_1,\, r_2,\, f_1,\, f_2 \in C^2[0, +\infty); & \\ \forall u \in (0, +\infty) \quad r_1'^2 + r_2'^2 + r_1^2 f_1'^2 + r_2^2 f_2'^2 = g, & (3) \\ \qquad\qquad\qquad\quad \gamma_1^2 r_1^2 + \gamma_2^2 r_2^2 = g u^2, & (4) \\ \qquad\qquad\qquad\quad \gamma_1 r_1^2 f_1' + \gamma_2 r_2^2 f_2' = 0; & (5) \\ \forall u \in [0, +\infty) \quad (r_1(u) \geqslant 0 \bigwedge r_2(u) \geqslant 0). & \end{cases}$$

If in chart 1 we pass to the Cartesian coordinates $\{x, y\}$, then $x^1(x, y)$, $x^2(x, y)$, $x^3(x, y)$, $x^4(x, y) \in C^2(R^2)$, and the same is true when passing to the Cartesian coordinates in chart 2. If the immersion exists, problem $\{1\}$ possesses a solution.

It is clear that when problem $\{1\}$ is solvable, formulas (2) give the desired immersion. For the immersion we are going to construct the plane (X^1OX^2) will turn out to be tangent to the resulting surface at the point O_1. Let r_1, r_2, f_1, f_2, be a solution to problem $\{1\}$. Then for all $u \in \{x : r_1^2(x) \neq gx^2/\gamma_1^2\}$ the relations

$$r_2 = \frac{\sqrt{gu^2 - \gamma_1^2 r_1^2}}{|\gamma_2|}, \tag{6}$$

$$r_2' = \frac{(gu^2)' - 2\gamma_1^2 r_1 r_1'}{2|\gamma_2|\sqrt{gu^2 - \gamma_1^2 r_1^2}} \tag{7}$$

hold. From (3)–(7) we obtain

$$\left(\varepsilon_0\sqrt{\gamma_2^2(gu^2 - \gamma_1^2 r_1^2)(-4a\varphi - (gu^2)'^2)} + (gu^2)'\gamma_1^2 r_1\right)/2a = r_1',$$

where

$$a = \gamma_2^2(gu^2 - \gamma_1^2 r_1^2) + \gamma_1^4 r_1^2; \quad k = f_1'^2; \quad \varphi = r_1^2\{1 + \gamma_1^2 r_1^2/(gu^2 - \gamma_1^2 r_1^2)\}k - g;$$
$$\forall u(|\varepsilon_0(u)| = 1).$$

We introduce the change of variable $\xi_1 = r_1|\gamma_1|/\sqrt{g}\,u$. Denote $\alpha = ug'/2g$. Then

$$\xi_1' u = \left\{ \varepsilon_0|\gamma_1||\gamma_2|\sqrt{(1-\xi_1^2)\left(b\left[1 - \frac{ku^2\xi_1^2}{\gamma_1^2(1-\xi_1^2)}\right] - (\alpha+1)^2\right)} \right.$$
$$\left. + (\alpha+1)\xi_1(\gamma_1^2 - \gamma_2^2)(1-\xi_1^2) \right\}/b, \qquad 0 \leqslant \xi_1 \leqslant 1, \tag{8}$$

where $b = \gamma_2^2(1-\xi_1^2) + \gamma_1^2\xi_1^2$. Let, for instance, $\gamma_2^2 \leqslant \gamma_1^2$. Let $\exists u(\gamma_1^2 < (\alpha(u)+1)^2)$, then at the point u relation (8) cannot be fulfilled because $b \leqslant \gamma_1^2 < (\alpha+1)^2$ and the radicand is negative. If at the point u $r_1^2 = gu^2/\gamma_1^2$, this procedure can be repeated for r_2. Therefore, when problem $\{1\}$ is solvable, we have $(\alpha+1)^2 \leqslant \gamma_1^2$.

We set the following problem:

$$\{2\}\quad \begin{cases} \xi_1 \in C^2[0,\, u_2], \qquad k \in C^1[0,\, u_2]; \\ \forall u \in (0,\, u_2)\ \xi_1' u = \Bigg\{ (\alpha+1)\xi_1(1-\xi_1^2)(\gamma_1^2-1) \\ \quad -\varepsilon_0\gamma_1\sqrt{(1-\xi_1^2)\Big(b\Big[1-\dfrac{ku^2}{\gamma_1^2}\dfrac{\xi_1^2}{1-\xi_1^2}\Big]-(\alpha+1)^2\Big)}\ \Bigg\}\dfrac{1}{b}; \\ \forall u \in [0,\, u_2]\ (0 \leqslant \xi_1(u) < 1 \bigwedge k(u) \geqslant 0), \\ \text{where } \gamma_1 \geqslant 2; \quad |\varepsilon_0| = 1; \quad \operatorname{sign}\varepsilon_0 = \operatorname{sign}(\alpha+1); \\ \xi_1(u_2) = \xi_0, \text{ where } \xi_0 \in [\overline{\xi}_0,\, 1), \quad b = 1+(\gamma_1^2-1)\xi_1^2. \end{cases}$$

Introduce the notation

$$a = 1 - \frac{ku^2}{\gamma_1^2}\frac{\xi_1^2}{1-\xi_1^2}.$$

We now show that if $\gamma_1^2 > (\alpha+1)^2$, then for any u_2 there exists $\overline{\xi}_0$ such that problem {2} possesses a solution.

Take some $u_1 \in (0,\, u_2)$. We set $\xi_1 = \beta_1 u + \beta_3 u^3$ on $[0,\, u_1]$. Denote $\xi_1' u = f$. Then if there is a solution, k can be expressed in terms of ξ_1 and f:

$$k = \frac{\gamma_1^2}{b\xi_1^2u^2}\Big\{ b(1-\xi_1^2) - f^2\frac{b^2}{\gamma_1^2} + 2f\xi_1(1-\xi_1^2)(\alpha+1)\frac{(\gamma_1^2-1)}{\gamma_1^2}b \\ - \xi_1^2(1-\xi_1^2)^2(\alpha+1)^2\frac{(\gamma_1^2-1)^2}{\gamma_1^2} - (\alpha+1)^2(1-\xi_1^2)\Big\}. \tag{9}$$

We expand α by Taylor's formula and then substitute it into (9) together with the expression for ξ_1. The coefficient in the term of the order of u^2 for the expression in curly brackets is $\beta_1^2(3-4/\gamma_1^2) - \alpha''(0)$, and there are no terms of the 0th, 1st, and 3rd orders. Put $\beta_1 = \sqrt{\alpha''(0)/(3-4/\gamma_1^2)}$. Then using (9) we obtain $k \in C^1[0,\, u_1]$. The terms of the order of u^4 are $2\beta_1\beta_3(\gamma_1^2+3-7/\gamma_1^2)+R$, where R symbolizes the terms not containing β_3. Since $\gamma_1 \geqslant 2$, we have $\gamma_1^2 + 3 - \dfrac{7}{\gamma_1^2} > 0$, and therefore the following assertion holds: $\forall\gamma_1\exists u_1\exists\beta_3\forall u \in [0,\, u_1]\ (k > 0)$. Therefore $\sqrt{k} \in C^1[0,\, u_1]$. It can easily be seen that

$$\forall\gamma_1\forall\beta_3\exists u_1\forall u \in (0,\, u_1](f - \xi_1(1-\xi_1^2)(\alpha+1)(\gamma_1^2-1)/b < 0). \tag{10}$$

In this case (9) implies that the equation in problem {2} turns into an identity upon the substitution of $\xi_1 = \beta_1 u + \beta_3 u^3$ and the resulting k, and hence these two functions are in fact a solution to problem {2} on $[0,\, u_1]$ (without boundary condition).

We take $\delta(u) \in C^2[0,\, u_2]$ and

$$u\xi_1' = (\alpha+1)\xi_1(1-\xi_1^2)(\gamma_1^2-1-\delta)/b. \tag{11}$$

If there is a smooth solution of class C^2 to (11), then the relations

$$\frac{\delta(\alpha+1)\xi_1(1-\xi_1^2)}{b} = \frac{\varepsilon_0\gamma_1\sqrt{(1-\xi_1^2)(ab-(\alpha+1)^2)}}{b}, \tag{12}$$

$$\delta^2(\alpha+1)^2\xi_1^2\frac{(1-\xi_1^2)}{\gamma_1^2} = ab - (\alpha+1)^2 \tag{13}$$

must hold. If $\delta(u) \geqslant 0$, then, by virtue of the choice of ε_0, relation (13) implies (12). We express k:

$$k = \frac{\gamma_1^2(1-\xi_1^2)}{bu^2\xi_1^2}\Big\{ 1 + \xi_1^2\Big(\gamma_1^2 - 1 - \frac{(\alpha+1)^2}{\gamma_1^2}(1-\xi_1^2)\delta^2\Big) - (\alpha+1)^2 \Big\}. \tag{14}$$

Let $\delta \equiv \gamma_1^2 - 1$ on $[u_1, u_2]$. Since $\gamma_1^2 > (\alpha + 1)^2$, we have

$$\exists \overline{\xi}_0 \in (0, 1) \forall \xi_0 \in [\overline{\xi}_0, 1) \forall u \in [u_1, u_2] (k(u) > 0), \quad \xi_1 \equiv \xi_0.$$

Consequently, ξ_0, k is in fact a solution to problem $\{2\}$ on $[u_1, u_2]$. Consider some $u_1^* \in (0, u_1)$ and $\delta(u)$ such that $0 \leqslant \delta \leqslant \gamma_1^2 - 1$ on $[u_1^*, u_1]$. There is u_1^* such that for any δ satisfying the indicated conditions there exists a solution to (11) on $[u_1^*, u_1]$. Therefore $u\xi_1' \leqslant (\alpha+1)\xi_1(1-\xi_1^2)(\gamma_1^2-1)/b$. If we perform the change of variable $t = -u$ and apply Chaplygin's lemma, we find that $\xi_1 \geqslant \overline{\xi}_1$ for all δ, where $u\overline{\xi}_1 = (\alpha+1)\overline{\xi}_1(1-\overline{\xi}_1^2)(\gamma_1^2-1)/b$. Then $\exists u_1^* \forall \delta \forall u \in [u_1^*, u_1] (k(u) > 0)$. We set $\delta(u) \equiv 0$ on $[0, u_1^*]$. We can assume that u_1 is so small that $1 < \alpha+1 < 2$. In this case in the domain of the solution to Eq. (11) on $(0, u_1^*]$ the relations $u\xi_1' \leqslant 2(\gamma_1^2 - 1)\xi_1$ and

$$\xi_1 \geqslant Cu^{2(\gamma_1^2-1)}; \qquad u\xi_1' \geqslant \xi_1 \frac{(1 - \xi_1^2(u_1^*))(\gamma_1^2 - 1)}{1 + (\gamma_1^2 - 1)\xi_1^2(u_1^*)};$$

$$\xi_1 \leqslant \widetilde{C} u^{\frac{(\gamma_1^2-1)(1-\xi_1^{*2})}{1+(\gamma_1^2-1)\xi_1^{*2}}}$$

must hold. Then the solution exists on $(0, u_1^*]$, and we have $\xi_1 \to 0 + 0$ for $u \to 0 + 0$, whence $\exists \widetilde{u} \in (0, u_1^*]((\gamma_1^2 - 1)(1 - \xi_1^2)/[1 + (\gamma_1^2 - 1)\xi_1^2] > 1)$, and, consequently, $\xi_1 \leqslant \overline{C}u^{1+\varepsilon}$ on $(0, \widetilde{u}]$, where $\varepsilon > 0$. Therefore $\exists \overline{u} \in (0, u_1^*](k(\overline{u}) = 0 \bigwedge \forall u \in (\overline{u}, u_1^*](k > 0))$, which follows from (14), and ξ_1, k is in fact a solution to problem $\{2\}$ on $(\overline{u}, u_1^*]$. However, we already have a solution of the form of $\beta_1 u + \beta_3 u^3$ on $[0, u_1^*]$, and it can also be represented as a solution to (11), relation (12) implying that the corresponding value is $\delta \geqslant 0$. Therefore (14) implies that $\xi_1(\overline{u}) < \beta_1\overline{u} + \beta_3\overline{u}^3$. Hence, $\exists u_0 \in (\overline{u}, u_1^*](\xi_1(u_0) = \beta_1 u_0 + \beta_3 u_0^3)$. We introduce the following notation:

$$\xi_1^{(1)} = \beta_1 u + \beta_3 u^3; \quad \xi_1^{(2)} = \xi_1;$$

$$\xi_1 = \begin{cases} \xi_1^{(1)} & \text{on} \quad [0, u_0], \\ \xi_1^{(2)} & \text{on} \quad [u_0, u_2]. \end{cases}$$

Generally speaking, ξ_1' and k undergo jump at the point u_0. Let, first, $\xi_{10}^{(2)'} > \xi_{10}^{(1)'}$. We can show that for any $\varepsilon > 0$ there exist a number $\mu > 0$ and a function $\Delta(u)$ such that $\Delta \equiv 1$ outside $[u_0 - \mu, u_0 + \mu]$, $\|\Delta - 1\|_{C[0, u_2]} \leqslant \varepsilon$, $\xi_1^{(1)'} \leqslant (\Delta\xi_1)' \leqslant \xi_1^{(2)'}$ on $[u_0 - \mu, u_0 + \mu]$, and $\Delta\xi_1 \in C^2[0, u_2]$. It follows from (9) that $k[\Delta\xi_1] \in C^1[0, u_2]$. If ε is sufficiently small, then, since (9) involves a quadratic trinomial with the variable f with a negative first coefficient, we conclude that there are μ and Δ such that $k[\Delta\xi_1] > 0$. We do the same in the case $\xi_1^{(2)'} < \xi_1^{(1)'}$. If $\xi_{10}^{(2)'} = \xi_{10}^{(1)'}$, then we require that $\|\Delta - 1\|_{C^1} \leqslant \varepsilon$, which also leads to the smoothing. Denote $\xi_1 = \Delta\xi_1$, $k(u) = k[\Delta\xi_1](u)$. It is clear that ξ_1, k is a solution to problem $\{2\}$ (the condition similar to (10) is also readily verified).

Set

$$f_1 = \int_0^u \sqrt{k}\, dp; \qquad \xi_2 = \sqrt{1 - \xi_1^2}; \qquad f_2 = \int_0^u \frac{\xi_1^2}{\gamma_1^2\xi_2^2}\sqrt{k}\, dp.$$

Then we have r_1, r_2, f_1, $f_2 \in C^2[0, u_2]$, which means that there exists a solution to problem $\{1\}$ on $[0, u_2]$. We make change of variable $\widetilde{u} = 1/u$, $\widetilde{\xi}_1(\widetilde{u}) = \xi_1(1/\widetilde{u})$ in problem $\{2\}$ (this transformation law follows from the definition of ξ_1). We can show that $\widetilde{\xi}_1$ satisfies the equation

$$\widetilde{u}\widetilde{\xi}_1' = \frac{1}{b}\left\{ \varepsilon_0\gamma_1\sqrt{(1 - \widetilde{\xi}_1^2)(ab - (\widetilde{\alpha} + 1)^2)} + \widetilde{\xi}_1(1 - \widetilde{\xi}_1^2)(\widetilde{\alpha} + 1)(\gamma_1^2 - 1) \right\}$$

on $[1/u_2, +\infty)$. We can assume that u_2 is so large that $\forall u \geqslant u_2/2$ $(\alpha + 1 \leqslant 0)$ ($\alpha \to -2$ for $u \to +\infty$, which follows from the Gauss-Bonnet theorem). Then $\varepsilon_0(u) = -1$, and $\widetilde{\xi}_1$ satisfies an equation of the same form on $[1/u_2, 2/u_2]$ as the one for ξ_1 on $[0, u_2]$. We can similarly construct $\widetilde{\xi}_1$ on $[0, 1/u_2]$, and in this case we have $\widetilde{\xi}_1 \in C^2[0, +\infty)$. We now investigate the smoothness at the poles in the Cartesian coordinates.*)

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*) Here we use an idea in [1].

Denote $z = x + iy \in Q$; $x^1 + ix^2 = \{Cu^2 + o(u^2)\}\exp\{i\gamma_1 t\} = o(|z|)$; $x^1 + ix^2 \in C^1(R^2)$; if $\gamma_1 = 2$, then $x^1 + ix^2 = Cz^2 + o(|z|^2)$. Therefore $x^1 + ix^2 \in C^2(R^2)$. If we are interested in the class $\overline{C}^1(R^2)$, then $\exists\gamma_1\,(\forall u \in [0,\, u_2]((\alpha+1)^2 < \gamma_1^2) \bigwedge \forall u \in [0,\, 1/u_2]((\tilde{\alpha}+1)^2 < \gamma_1^2))$. Furthermore, $x^3 + ix^4 = \sqrt{g(0)}\,z + o(|z|^2)$, and, consequently, $x^3 + ix^4 \in C^2(R^2)$. The immersion equations $g_{nm}\dfrac{\partial x^n}{\partial u^\alpha}\dfrac{\partial x^m}{\partial u^\beta} = G_{\alpha\beta}$ imply that the rank of the mapping F is maximal. Let us show that the condition $(\alpha + 1)^2 \leqslant 4$ is necessary in searching for a solution of class $C^2(M)$. It follows from the equations of problem $\{1\}$ that $r_k'(0) \neq 0$ for some k. For example, let $r_2'(0) \neq 0$. Then $x^3 + i\ \mathrm{sign}\,\gamma_2 x^4 = r_2\exp\{i|\gamma_2|t + f_2\} = Cu\exp\{i|\gamma_2|t\} + o(|z|)$, where $C \in Q$, and, since $x^3 + i\ \mathrm{sign}\,\gamma_2 x^4 \in C^1(R^2)$, we have $|\gamma_2| = 1$. We replace r_1 by ξ_1 directly in problem $\{1\}$. This yields

$$\xi_1^2 + \xi_2^2 = 1, \quad \frac{1}{\gamma_1^2}\{u\xi_1' + \xi_1(\alpha+1)\}^2 + \frac{1}{\gamma_2^2}\{u\xi_2' + \xi_2(\alpha+1)\}^2 + \frac{ku^2}{\gamma_1^2}\xi_1^2\Big\{\,1 + \frac{\xi_1^2}{\xi_2^2}\,\Big\} = 1.$$

For $u = 0$ we have $\xi_1^2/\gamma_1^2 + \xi_2^2 = \xi_1^2 + \xi_2^2$. Since $K(0) < 0$, we have $\exists\varepsilon > 0 \quad \forall u \in (0,\, \varepsilon)$ $(\alpha > 0)$, and, consequently, $\gamma_1^2 > 1$ (otherwise $\exists u((\alpha + 1)^2 > \max(\gamma_1^2,\, \gamma_2^2))$). Therefore $\xi_1(0) = 0$, which means that there is $\varepsilon > 0$ such that ξ_1 satisfies the equation of problem $\{2\}$ (the sign of ε_0 is unimportant) for all u belonging to $(0,\, \varepsilon]$. The fact that $r_1 \in C^2[0,\, +\infty)$ implies that $\xi_1 \in C^1(0,\, +\infty)$. Using the expansion $\xi_1 = \xi_1'(0)u + o(u)$ and repeating the calculations for k we obtain $\xi_1'(0) > 0$. Therefore $x^1 + i\ \mathrm{sign}\,\gamma_1 x^2 = \{Cu^2 + o(|z|^2)\}\exp\{i|\gamma_1|t\}$ and $|\gamma_1| = 2$ but $(\alpha + 1)^2 \leqslant \gamma_1^2 = 4$.

Remark. The arbitrariness in the choice of Δ means that the constructed surface is deformable in the class of surfaces of revolution.

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