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MODERN METHODS OF PERTURBATION THEORY FOR CALCULATING HYDROACOUSTIC FIELDS

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The perturbation theory methods for approximate solution of the Helmholtz equation governing the waveguide ocean model are discussed. Nonlinear perturbation theory, or the delinearization method, is introduced in addition to the standard perturbation theory modified in accord with the specificity of the acoustic problems. The new theory does not require the knowledge of the whole eigenvalue spectrum of the unperturbed problem and, besides, all the corrections are written explicitly in quadratures. Unlike the existing theories the approach suggested allows one to avoid nonremovable singularities in basic formulae for any waveguide types by choosing appropriate wavefunction representation in the form of polynomials.

Applied investigations of ocean by acoustic methods require elaboration of new algorithms for calculating acoustic fields. A frequently used model representing ocean in the form of a nonuniform planar waveguide with certain boundary conditions leads to the Helmholtz equation [1]

$$\psi_l''(z) + k^2(z)\psi_l(z) = \varkappa_l^2\psi_l(z), \tag{1}$$

where $\varkappa_l^2$ is the lth eigenvalue, a quantity having a meaning of the horizontal wavenumber squared, ψ_l is the corresponding waveguide eigenfunction (mode), $k(z) = \omega/c(z)$, and $c(z)$ is the acoustic velocity profile in the waveguide. The field in the waveguide at the depth z and distance r from the point source is written in the form of an expansion in terms of the eigenfunctions

$$U(r,z) = \sum_{l=1}^{\infty} \psi_l(z_0)\psi_l(z)H_0^{(1)}(\varkappa_l r), \tag{2}$$

where $H_0^{(1)}(\varkappa_l r)$ is Hankel's function of the first kind. At large distances from the source of sound Hankel's function can be replaced by its asymptotic representation. This reduces Eq. (2) to

$$U(r,z) = \sum_{l=1}^{\infty} \sqrt{2/\pi\varkappa_l r}\,\psi_l(z_0)\psi_l(z)\exp\{i(\varkappa_l r - \pi/4)\}.$$

Direct calculations of the acoustic field by this equation presents difficulties since the form of eigenfunctions $\psi_l(z)$ more often is unknown. This is associated with the fact that Eq. (1) has the exact solution only in a number of cases. To calculate the fields one has to elaborate approximate methods.

WKB, Feynman's diagram, and variational methods can serve as examples of such techniques. However the use of these methods in hydroacoustics is rather inefficient, therefore they have not become widespread. Thus, the WKB method requires small variations of the properties of the medium over the wavelength, which does not hold for low-frequency perturbations. Application of Feynman's diagrams involves great difficulties associated with the calculation of multiloop diagrams, besides, the series obtained do not always converge. The variational methods are inconvenient because criteria of the accuracy of the results obtained are lacking, while the problem of the trial-function choice is usually solved intuitively.

One of the approximate approaches most widely spread in quantum mechanics, the theory of oscillations, and other areas in physics is the perturbation theory technique. It is employed when the wavenumber squared $k^2(z)$ can be written as a sum of two terms

$$k^2(z) = k_0^2(z) + k_1^2(z).$$

The wavenumber profile (i. e., actually, the acoustic velocity profile) $k_0(z)$ is selected such that the Helmholtz equation

$$\psi_l''(z) + k_0^2(z)\psi_l(z) = \varkappa_l^2\psi_l(z), \tag{3}$$

referred to below as an "nonperturbed equation", have an exact solution, while perturbation $k_1^2(z)$ result in small corrections to the solution of the nonperturbed equation.

Successive calculation of these corrections (the first, second, and so on approximations) yields, as a rule, an expansion in terms of some formal parameter λ.

We consider two versions of perturbation theory. In the first one, referred to as "the standard perturbation theory", because of its extensive use in various fields of the wave physics, corrections are sought for directly to the solution of Eq. (1). In the second approach, developed in recent years, and called "nonlinear perturbation theory", or "delinearization method", Eq. (1) is at first transformed into a nonlinear first-order differential equation, and then corrections to its solution are calculated.

1. THE STANDARD PERTURBATION THEORY

Detailed description of this theory can be found in a number of textbooks on quantum mechanics (see, e. g., [2]). As applied to hydroacoustic problems, the basic formulae have some distinctions associated with physical peculiarities of sound propagation in ocean.

We rewrite Eq. (1) as follows

$$(\varkappa_l^2 - k_0^2(z) - k_1^2(z))\psi_l(z) = 0. \tag{4}$$

Now the problem is to find from this equation approximate values of $\varkappa_l^2$ and waveguide eigenfunctions $\psi_l(z)$ corresponding to them with due regard for a correction to the wavenumber profile $k_1^2(z)$ (i. e., taking into account hydrological perturbations). The solution is sought in the form of series

$$\begin{aligned} \psi_l &= \psi_l^0 + \lambda\psi_l^1 + \lambda^2\psi_l^2 + \dots, \\ \varkappa_l^2 &= \varkappa_{0l}^2 + \lambda\varkappa_{1l}^2 + \lambda^2\varkappa_{2l}^2 + \dots, \end{aligned} \tag{5}$$

where $\lambda\psi_l^1$ and $\lambda\varkappa_{1l}^2$ are quantities of the first order of smallness as compared to ψ_l^0 and $\varkappa_{0l}^2$, respectively, $\lambda^2\psi_l^2$ and $\lambda^2\varkappa_{2l}^2$ are the terms of the second order of smallness, etc. Substitution of Eqs. (5) and (4) and isolation of the zero-order terms in λ yields

$$(\varkappa_{0l}^2 - k_0^2(z))\psi_l^0(z) = 0, \tag{6}$$

which allows one to find, to the zero approximation, all the eigenvalues $\varkappa_{01}^2$, $\varkappa_{02}^2, \dots$ and eigenfunctions corresponding to them. In terms of the first order in λ the equation of the first-order perturbation theory approximation reads

$$(\varkappa_{0l}^2 - k_0^2(z))\psi_l^1(z) = (-\varkappa_{1l}^2 - k_1^2(z))\psi_l^0(z). \tag{7}$$

Let the unperturbed modes constitute a complete set of orthonormalized functions. Then the solution for $\psi_l^1(z)$ can be written as an expansion in terms of these functions:

$$\psi_l^1(z) = \sum_k C_{kl}\psi_k^0(z). \tag{8}$$

Substituting this formula into Eq. (7) yields, with due regard for Eq. (6),

$$\sum_k C_{kl}(\varkappa_{0l}^2 - \varkappa_{0k}^2)\psi_k^0(z) = (-\varkappa_{1l}^2 - k_1^2(z))\psi_l^0(z). \tag{9}$$

Multiplying both sides of Eq. (8) by ψ_l^0 and integrating over the waveguide depth gives the sought correction to the eigenvalue:

$$\varkappa_{1l}^2 = \int_0^H k_1^2(z)\psi_l^{0^2}(z)\,dz. \tag{10}$$

To find coefficients C_{kl} in Eq. (8) one has to use Eq. (9) multiplied by $\psi_m^0(z)$ and integrated over the waveguide depth. Then Eq. (8) for the lth mode perturbation assumes the form

$$\psi_l^1(z) = \sum_{m\neq l} \frac{\psi_m^0(z)}{\varkappa_{0l}^2 - \varkappa_{0m}^2} \int\limits_0^H \psi_m^0(z) k_1^2(z) \psi_l^0(z)\,dz. \tag{11}$$

Since in applied hydroacoustic problems specification of the acoustic velocity profile throughout the waveguide depth seems hardly possible, one has to introduce an expansion in terms of some function basis using the sampling theorem

$$k_1^2(z) = \sum_{j=1}^{N} k_{1j}^2 \Theta_j(z),$$

where N is the number of samples. Introducing designation

$$\int\limits_0^H \psi_l^0(z)\Theta_j(z)\psi_m^0(z)\,dz = B_{lmj},$$

we transform Eqs. (10) and (11) to

$$\varkappa_{1l}^2 = \sum_{j=1}^{N} k_{1j}^2 B_{llj},$$

$$\psi_l^1 = \sum_{j=1}^{N} k_{1j}^2 \sum_{m\neq n} \frac{\psi_m^0(z)}{\varkappa_{0l}^2 - \varkappa_{0m}^2} B_{lmj}.$$

We denote the acoustic field in the waveguide with no perturbation by $U_0(r,z)$. It is calculated by Eq. (2) in terms of the nonperturbed eigenfunctions. A change of the wavenumber profile for $k_1^2(z)$ results in an acoustic-field perturbation proportional to it.

$$\begin{aligned}\Delta U(r,z) = \sum_{j=1}^{N} k_{1j}^2 \Big[\sum_{l=1}^{\infty} &\sqrt{2/\pi\varkappa_{0l} r}\exp\{i(\varkappa_{0l} r - \pi/4)\} \\ &\times \sum_{m\neq l} \frac{B_{lmj}}{\varkappa_{0l}^2 - \varkappa_{0m}^2} (\psi_l^0(z)\psi_m^0(z_0) + \psi_l^0(z_0)\psi_m^0(z)) \\ &+ \psi_l^0(z_0)\psi_l^0(z)\Big(\frac{ir}{2\varkappa_{0l}} B_{mmj}\Big)\Big].\end{aligned} \tag{12}$$

Denoting the sum in the curly brackets by $Q_j(r,z)$ the above equation can be rewritten in the following form

$$\Delta U(r,z) = \sum_{j=1}^{N} k_{1j}^2 Q_j(r,z). \tag{13}$$

Hence, a relation is deduced which allows one to find acoustic-field perturbations pertaining to wavenumber perturbations preset at some horizons, using the elements of matrix Q evaluated from the formulae for the first perturbation theory corrections as the calculation coefficients. Note, however, that in practice one has to solve the inverse problem, i. e., to calculate changes in the hydrological profile at sites where hydrophones are positioned for measured acoustic fields. Hence, Eq. (13) is transformed to a set of linear equations with the number of unknowns equal to that of hydrophones. Solving it one can recover the acoustic velocity profile along the waveguide.

Let us dwell in more detail on the analysis of Eq. (12). Summation over all the modes is unrealistic. There is no necessity to do this, since peculiarities of sound wave propagation in ocean waveguides allow one to obtain fairly good results using only a finite number of lower modes in numerical calculations. However, the presence of the second sum in the formula necessitates the knowledge of the other nonperturbed modes (i. e., the whole spectrum of the nonperturbed problem) when computing corrections of each mode. Besides, a rule for summing up the series in the calculation formula should be formulated, and this necessitates additional physical investigations in each particular case. Diverging perturbation theory series are quite typical. At the same time, the simplicity of perturbation theory formalism, against shortcomings of other methods for solving approximately the Helmholtz-type equations, which has been discussed above, makes it efficient in solving many hydroacoustic problems.

2. THE DELINEARIZATION METHOD

The above-mentioned drawbacks of the standard theory led the investigators to elaborating a new perturbation theory modification. Its foundations were laid in [3] where the theory was described for the ground and first excited states; then they were extended to three-dimensional space [4]. The new theory was presented, presumably in more completed form, in [5] and [6]. But the presence of singularities in the basic equations in these papers makes impossible computational solution of the problems using perturbation theory. The nonlinear perturbation theory as applied to hydroacoustic problems (the "delinearization method") is presented in [7] and [8], where, however, the aforementioned singularities were eliminated only in some particular cases. In the theory version suggested we managed to avoid this difficulty by selecting a specific (polynomial) wavefunction representation.

We separate individual amplitude and phase parts in the lth eigenfunction representing the solution to the Helmholtz equation with given boundary conditions and an acoustic velocity profile:

$$\psi_l(z) = [\prod_{i=1}^{l}(z-\alpha_{il})]\exp\{-\varphi_l(z)\},$$

where the amplitude part is represented in the form of an l-order polynomial, α_{il} $(i = 1, \dots, l)$ are the roots of the polynomial corresponding to the positions of the nodal points of the wavefunction $\psi_l(z)$, and the phase function $\varphi_l(z)$ has no singularities. The nonperturbed mode is written similarly

$$\psi_l^0(z) = [\prod_{i=1}^{l}(z-\alpha_{il}^0)]\exp\{-\varphi_l^0(z)\}.$$

As in the standard perturbation theory (5), the eigenvalues and eigenfunctions of the waveguide are expanded into a series in powers of the formal parameter λ. Positions of eigenfunction zeros, the phase function, and its derivative are written in a similar form:

$$\begin{aligned}\alpha_{il} &= \alpha_{il}^0 + \lambda\alpha_{il}^1 + \lambda^2\alpha_{il}^2 + \dots\,,\\ \varphi_l(z) &= \varphi_l^0(z) + \lambda\varphi_l^1(z) + \lambda^2\varphi_l^2(z) + \dots\,,\\ g_l(z) &= g_l^0(z) + \lambda g_l^1(z) + \lambda^2 g_l^2(z) + \dots\,.\end{aligned}$$

Substituting the above-written equations into Helmholtz equation (1) and isolating the terms with λ in the zero power yields

$$\begin{aligned}&(g_l^{0^2}(z) - dg_l^0(z)/dz)\prod_{i=1}^{l}(z-\alpha_{il}^0) - 2g_l^0(z)\sum_{i=1}^{l}\Big\{\prod_{\substack{\kappa=1\\ \kappa\neq i}}^{l}(z-\alpha_{\kappa l}^0)\Big\}\\ &+2\sum_{C_l^{l-1}}\Big\{\prod_{\substack{m=1\\ \kappa_m=1,\dots,N}}^{l-2}(z-\alpha_{\kappa_m l}^0)\Big\} = (\varkappa_{0l}^2 - k_0^2(z))\prod_{i=1}^{l}(z-\alpha_{il}^0).\end{aligned}$$

The equation with the first-order terms in λ reads

$$
\begin{aligned}
&(2g_l^0(z)g_l^1(z) - dg_l^1(z)/dz)\prod_{i=1}^{l}(z-\alpha_{il}^0) - 2g_l^1(z)\sum_{i=1}^{l}\Big\{\prod_{\substack{\kappa=1\\ \kappa\neq i}}^{l}(z-\alpha_{\kappa l}^0)\Big\}\\
&\quad - (g_l^{0^2}(z) - dg_l^0(z)/dz)\sum_{i=1}^{l}\Big\{\alpha_{il}^1\prod_{\substack{\kappa=1\\ \kappa\neq i}}^{l}(z-\alpha_{\kappa l}^0)\Big\} + 2g_l^0(z)\sum_{i=1}^{l-1}\alpha_{il}^1\prod_{\substack{m=1\\ \kappa_m\neq i}}^{l-2}(z-\alpha_{\kappa l}^0)\\
&\quad - 2\sum_{i=1}^{l}\alpha_{il}^1\Big\{\prod_{\substack{m=1\\ \kappa_m\neq i}}^{l-3}(z-\alpha_{\kappa_m l}^0)\Big\} = (\varkappa_{1l}^2 - k_1^2(z))\prod_{i=1}^{l}(z-\alpha_{il}^0)\\
&\quad - (\varkappa_{0l}^2 - k_0^2(z))\sum_{\substack{i=1\\ \kappa\neq i}}^{l}(z-\alpha_{\kappa l}^0).
\end{aligned}
$$

Multiplying all the terms of the first equation by $\prod_{i=1}^{l}(z-\alpha_{il}^0)\exp\{-2\varphi_l^0(z)\}$ and using the second equation we obtain

$$
\begin{aligned}
\frac{d\{-g_l^1\psi_l^{0^2}(z)\}}{dz} &= (\varkappa_{1l}^2 - k_1^2(z))\psi_l^{0^2}(z)\\
&\quad - \sum_{i=1}^{l}\alpha_{il}^1\frac{d}{dz}\Big[\Big(\prod_{\substack{\kappa=1\\ \kappa\neq i}}^{l}(z-\alpha_\kappa^0)^2\Big)\exp\{-2\varphi_l^0(z)\}\Big].
\end{aligned}
\tag{14}
$$

This is the main equation of the method considered. It is equivalent to Helmholtz equation (1) to within first-order terms in λ. But unlike Eq. (1), formula (14) is a nonlinear first-order equation, which has lent the name of "nonlinear perturbation theory" (or "delinearization method") to the method under consideration. All the basic formulae of the nonlinear perturbation theory are derived directly from Eq. (14). Thus, integration of its both sides over the waveguide depth from 0 to H yields the formulae for the first-order corrections to eigenvalues which are identical with the equations derived in the standard perturbation theory (the higher-order corrections differing from those of the standard theory). Corrections to the eigenfunction node positions are obtained by integrating Eq. (14) from 0 to α_i^0, and those to the phase function, by doubly integrating from 0 to z. After applying the sampling theorem the above expressions assume the following form

$$
\varkappa_{1l}^2 = \sum_{j=1}^{N} k_{1j}^2 B_{llj},
$$

$$
\alpha_{il}^1 = \sum_{j=1}^{N}\beta_{il}\int_0^{\alpha_i^0}(B_{llj} - \Theta_j(z))\psi_l^{0^2}(z)\,dz\cdot k_{1j}^2,
$$

$$
\begin{aligned}
\varphi_l^1(z) = \sum_{j=1}^{N}\int_0^{z}\frac{1}{\psi_l^{0^2}(z)}\Big\{&-\int_0^{z'}(B_{llj} - \Theta_j(z'))\psi_l^{0^2}(z')\,dz'\\
&+ \sum_{i=1}^{l}\beta_{il}\frac{\psi_l^{0^2}(z')}{(z'-\alpha_{il}^0)^2}\int_0^{\alpha_i^0}(B_{llj} - \Theta_j(z'))\psi_l^{0^2}(z')\,dz'\Big\}\,dz\cdot k_{1j}^2,
\end{aligned}
$$

where

$$
\beta_{il} = \exp\{2\varphi_l^0(\alpha_{il}^0)\}\Big/\prod_{\substack{\kappa=1\\ \kappa\neq i}}^{l}(\alpha_i^0 - \alpha_\kappa^0).
$$

Now, for a perturbation of the acoustic field caused by hydrological factor $k_1(z)$ we write

$$
\begin{aligned}
\Delta U(r,z) = \sum_{j=1}^{N} k_{1j}^2 \sum_{l=1}^{\infty} \sqrt{2/\pi \varkappa_{0l} r} \cdot \exp\{i(\varkappa_{0l} r - \pi/4)\} \psi_l^0(z_0) \psi_l^0(z) \\
\times \Big\{ i\frac{r_0}{2\varkappa_{0l}} B_{llj} - \sum_{i=1}^{l} \frac{1}{z-\alpha_i^0} \beta_{il} \int_0^{\alpha_i^0} (B_{llj} - \Theta_j(z)) \psi_l^{0^2}\, dz \\
- \sum_{i=1}^{l} \frac{1}{z_0 - \alpha_i^0} \beta_{il} \int_0^{\alpha_i^0} (B_{llj} - \Theta_j(z)) \psi_l^{0^2}(z)\, dz \\
+ \int_0^{z} \frac{1}{\psi_l^{0^2}(z)} \Big\{ \int_0^{z'} (B_{llj} - \Theta_j(z')) \psi_l^{0^2}(z')\, dz' \\
- \sum_{i=1}^{l} \beta_{il} \psi_l^{0^2}(z) \frac{1}{(z-\alpha_i^0)^2} \int_0^{\alpha_i^0} (B_{llj} - \Theta_j(z')) \psi_l^{0^2}(z')\, dz' \Big\}\, dz \\
+ \int_0^{z_0} \frac{1}{\psi_l^{0^2}(z)} \Big\{ \int_0^{z'} (B_{llj} - \Theta_j(z')) \psi_l^{0^2}(z')\, dz' \\
- \sum_{i=1}^{l} \beta_{il} \frac{1}{(z'-\alpha_i^0)^2} \int_0^{\alpha_i^0} (B_{llj} - \Theta_j(z')) \psi_l^{0^2}(z')\, dz' \Big\}\, dz \Big\}.
\end{aligned}
$$

Denoting, as in the standard theory, the inner sum by $Q(r,z)$ we arrive at Eq. (13) with the only difference that matrix Q is calculated in this case by formulae for corrections in the nonlinear perturbation theory that is free of the drawbacks mentioned in Section 2, i.e., does not necessitate the knowledge of the whole spectrum of the nonperturbed problem. And since there is no series in the correction formulae in this theory, the problem of the series-summation procedure is eliminated. Moreover, all the corrections are expressed explicitly in quadratures, which makes this method convenient in computations.

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