Moscow University Physics Bulletin
Vol. 47, No. 2, pp. 63-67, 1992

Vestnik Moskovskogo Universiteta. Fizika
UDC 521.1

POST-NEWTONIAN APPROXIMATION IN RELATIVISTIC GRAVITATION THEORY WITH ACCOUNT OF GALACTIC ROTATION

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It is shown how the account of noninertiality of a coordinate system undergoing galactic rotation affects the post-Newtonian physics of bodies in the solar system.

In the standard post-Newtonian consideration of gravitational effects in the solar system it is assumed that the coordinate system in use related to the center of the Sun (the heliocentric system) is inertial. In other words, the noninertial effects due to the motion of the solar system together with the other neighboring stars of the local system round the galaxy center is neglected (see, e.g., [1, 2]). We shall consider the problem of noninertiality assuming that the solar system moves as a whole in a circular orbit round the galaxy center according to Newton's law. Proceeding from this assumption we calculate the corrections to the post-Newtonian trajectories of the planets (regarding them in the post-Newtonian approximation as points) round the Sun and the corresponding observed phenomena. We note that the remark that it is necessary to take into account the noninertial effects when considering the post-Newtonian formalism in the solar system was made long ago (see, e.g., [3] and references there), but up till now no concrete calculations of the corresponding effects with the noninertiality taken into account has been mentioned in literature. To avoid misunderstanding we stress at the very beginning that the axes of the noninertial coordinate system we are using are not fixed with respect to the inertial coordinate system but undergo certain rotation (see below).

We introduce an inertial Cartesian coordinate system (X, Y, Z) with the origin at the galaxy center and the axis Z directed along the rotation vector of the solar system in the circular orbit.

We also consider a coordinate system (x, y, z) with the origin at the center of mass of the solar system (i.e., practically, at the center of the Sun) such that the direction of the axis z coincides with that of the axis Z, the axis x is directed along the radius of the solar system circular orbit, and the axis y goes along the tangent line to the circular orbit.

Let R_0 be the orbit radius and let M be the effective galaxy mass concentrated, in our consideration, at the origin of the inertial coordinate system. Then the angular velocity w of rotation in the circular orbit is given by the formula

$$w^2 = M/R_0^3.$$

The relationship between the coordinates (X, Y, Z) and (x, y, z) is found from geometrical considerations:

$$\begin{aligned} X &= (R_0 + x)\cos(wt) - y\sin(wt), \\ Y &= (R_0 + x)\sin(wt) + y\cos(wt), \\ Z &= z. \end{aligned} \tag{1}$$

We assume that the planets in the solar system move as test bodies along the geodesics of the resultant curved space-time, i.e., are described by the equations

$$G^i = \frac{d^2X^i}{ds^2} = -\Gamma^i_{mn}\frac{dX^m}{ds}\frac{dX^n}{ds}. \tag{2}$$

The metric g_{ij} in (2) is determined by both the gravitational field of the Sun moving in the circular orbit and the galaxy gravitational field and obeys the equations of the relativistic gravitation theory (RGT):

$$\begin{aligned} R_{ij} - g_{ij}R/2 &= 8\pi T_{ij}, \\ D_j(\sqrt{-g}\, g^{ij}) &= 0. \end{aligned} \tag{3}$$

Here D is the covariant derivative in the metric γ_{ij} of the planar space, $\gamma_{ij}(t, X, Y, Z) = \mathrm{diag}(1, -1, -1, -1)$.

In the post-Newtonian approximation the solution to Eqs. (3) is (here we concretize, for the problem under study, the general form of the RGT post-Newtonian metric obtained in [1]):

$$
\begin{aligned}
g_{00} &= 1 - 2m/r(1 - (2V^2 + (\mathbf{r}\mathbf{V}/r^2)/2) - 2M/R + 2(m/r)^2 + 2(M/R)^2, \\
g_{0\alpha} &= 4m\mathbf{V}_\alpha/r, \\
g_{\alpha\beta} &= \gamma_{\alpha\beta}(1 + 2m/r + 2M/R),
\end{aligned}
\tag{4}
$$

where m is the conserved mass, $\mathbf{r} = \mathbf{R} - \mathbf{R}_0$, $\mathbf{R}_0 = (R_0\cos(wt), R_0\sin(wt), 0)$, $\mathbf{R} = (X, Y, Z)$, $\mathbf{V} = (-R_0 w\sin(wt), R_0 w\cos(wt), 0)$, and the quantities involved in (4) have the following order of smallness [4]:

$$
m/r \approx 10^{-6} - 10^{-8}, \quad M/R \approx w^2R^2 \approx 10^{-6}, \quad r/R_0 \approx 10^{-9}. \tag{5}
$$

We substitute (4) into the equations of geodesics (2) and find the accelerations G^i, after which, on passing from the coordinates (X, Y, Z) to the coordinates (x, y, z) by formulas (1), we determine the accelerations $g^i = d^2x^i/ds^2 \times (dt/ds)^{-2}$ and, using the condition $wt \ll 1$ ($w \approx 10^{-15}$ s^{-1}), retain in g^i only the terms of the Newtonian (~ 1 cm/s^2) and post-Newtonian ($\sim 10^{-8}$ cm/s^2) order. Since in this case the relation $G^3 = 0$ turns to be valid for $z = 0$ and $dz/dt = 0$, the motion of the test bodies does not fall outside the plane $z = 0$. Therefore, in what follows we confine ourselves to the case of the motion in the plane $z = 0$.

Hence, we obtain

$$
\begin{aligned}
g_x &= \frac{xm}{r^3}\Big[-1 + \frac{4m}{r} + 6V_c^2 + 2V_cv_x + 3v_x^2 - v_y^2 + \frac{3}{2}V_c^2\frac{y^2}{r^2}\Big] \\
&\quad + \frac{ym}{r^3}[V_cv_x + 4v_xv_y] + \frac{2V_cv_y}{R_0}, \\
g_y &= \frac{ym}{r^3}\Big[-1 + \frac{4m}{r} + 6V_c^2 + 3V_cv_x + 3v_y^2 - v_x^2 + \frac{3}{2}V_c^2\frac{y^2}{r^2}\Big] \\
&\quad + \frac{xm}{r^3}[4v_xv_y] - \frac{2V_cv_x}{R_0}, \\
V_c &= wR_0, \qquad v_x = dx/dt, \qquad v_y = dy/dt.
\end{aligned}
\tag{6}
$$

Let us consider some consequences of relations (6).

The anomalous perihelion shift W of a planet moving in the Newtonian approximation in an elliptic orbit with eccentricity e, conserved angular momentum $h = (mp)^{1/2}$, and the orbit parameter p is found from the equation [5]

$$
\frac{dW}{dt} = -\frac{p}{he}A_\parallel\cos f + \frac{p+r}{he}A_\perp\sin f, \tag{7}
$$

where $A_\parallel$ and $A_\perp$ are, respectively, the acceleration components parallel and perpendicular to the tangent line to the trajectory of motion:

$$
\begin{aligned}
A_\parallel &= g_x\cos f + g_y\sin f, \\
A_\perp &= -g_x\sin f + g_y\cos f, \\
r &= p/(1 + e\cos f), \\
x &= r\cos f, \\
y &= r\sin f, \\
r^2\frac{df}{dt} &= h.
\end{aligned}
$$

The explicit expressions for $A_{\|}$ and $A_{\perp}$ are the following:

$$A_{\perp} = \frac{h\sin f}{p^4}\Big[4ehm(1+e\cos f)^3 - \frac{2ep}{R_0}V_c + mpV_c(1+e\cos f)^3\Big],$$

$$\begin{aligned} A_{\|} &= \cos^2 f\frac{m}{p^4}(1+e\cos f)^2[3e^2h^2 + 3ehpV_c + 3p^2V_c^2/2] \\ &+ 2\cos f\frac{hm}{p^3}V_c(1+e\cos f)^3 + \frac{(1+e\cos f)}{p^4} \\ &\times \Big[-3e^2h^2m(1+e\cos f) - 3ehmpV_c(1+e\cos f) + h^2m(1+e\cos f)^3 \\ &- 2\frac{hp^3V_c}{R_0} - 4m^*p^2(1+e\cos f)^2 - 6mp^2V_c^2(1+e\cos f) \\ &+ m^*p^2(1+e\cos f)\Big]. \end{aligned} \tag{8}$$

Substituting (6) into (7) we obtain

$$\begin{aligned} \frac{dW}{dt} &= \cos^3 f\Big(\frac{3e}{p} + \frac{3pV_c^2}{2eh^2} + \frac{3V_c}{h}\Big) + \cos^2 f\Big(-\frac{3mV_c(1+3\cos f)}{eh} - \frac{mV_c}{eh} \\ &+ \frac{2p^2V_c}{hR_0(1+e\cos f)^2} + \frac{2p^2V_c}{hR_0(1+e\cos f)^3} - \frac{4m^*(1+e\cos f)}{p} - \frac{4m^*}{p}\Big) \\ &+ \cos f\Big(-\frac{3em}{p} - \frac{2p^2V_c}{ehR_0(1+e\cos f)} - \frac{4m^2(1+e\cos f)}{eh^2} - \frac{6pV_c^2}{eh^2} + \frac{mp}{eh^2} \\ &+ \frac{m(1+e\cos f)^2}{ep} - \frac{3mV_c}{h}\Big) + \frac{mV_c(1+3\cos f)}{eh} + \frac{mV_c}{eh} \\ &- \frac{2p^2V_c}{hR_0(1+e\cos f)^2} - \frac{2p^2V_c}{hR_0(1+e\cos f)^3} + \frac{4m^*(1+e\cos f)}{p} + \frac{4m^*}{p} \end{aligned} \tag{9}$$

(here the mass $m^* = m(1 - 3/2V_c^2)$ is introduced).

The integration of (9) along the orbit of motion results in the following value of the anomalous perihelion shift with account of the galactic rotation of the solar system:

$$dW = \frac{m\pi}{p}\Big[6 + \frac{whp^2}{m^2}k\Big], \tag{10}$$

$$k = -\frac{2}{(1-e^2)^{3/2}}.$$

For $w = 0$ anomalous shift (10) coincides with the standard post-Newtonian value for the anomalous shift in the RGT [1] but differs from it when the rotation is taken into account. It is interesting to note that the nonzero integrated contribution to the shift additional to the anomalous shift (the second term in (10)) comes only from the Coriolis terms (the last terms in (6)), the other terms in (6) giving only a zero contribution to integrated expression (10).

The magnitude of the shift additional to the anomalous shift for the Mercury amounts to a quantity of the order of $-43'' \times 0.01 = -0.43''$ per century (we remind the reader that for the Mercury the accepted value of the anomalous shift amounts to $42.95 \pm 0.215''$ per century [1]).

To explain this result we can suggest the following ways.

As follows from the result obtained here, the standard formula for the anomalous shift is obtained in a coordinate system whose axes are rigidly fixed with respect to the "fixed" stars not belonging to our galaxy (i. e., in a "quasi-inertial" coordinate system x', y', z', where the effects related to the Coriolis force are absent, and for $wt \ll 1$ the forces in the equations of motion coincide with forces (6) except for the Coriolis forces). The derived formula (10) means that there is a new additional effect in the phenomenon of anomalous shift appearing owing to the employment of a noninertial coordinate system (x, y, z) whose axes x and y, by definition, rotate with respect to the axes X, Y. This effect also exists in the Newtonian theory (the rotation of the plane of oscillations of a vector with the angular velocity of rotation of the axes of a markedly noninertial coordinate system). Indeed, since the Newtonian period of elliptic motion is

$$T = 2\pi p^{3/2} m^{-1/2} (1 - e^2)^{-3/2},$$

the ratio of the additional shift

$$d\,dW = \pi k w p^{3/2} m^{-1/2}$$

to the period T is equal to

$$d\,dW/T = -w.$$

The surprising thing is that the account of post-Newtonian terms did not change this result.

We also note that, by virtue of the condition v_x, $v_y \gg V_c$, the inclusion of the corrections to the rectilinear propagation of the light ray in the plane $z = 0$ reduces in (6) to the account of terms proportional only to m, which results in

$$\begin{aligned} g_x &= -\frac{mx}{r^3} + \frac{mx}{r^3}(3v_x^2 - v_y^2) + 4v_x v_y \frac{my}{r^3}, \\ g_y &= -\frac{my}{r^3} + \frac{my}{r^3}(3v_y^2 - v_x^2) + 4v_x v_y \frac{mx}{r^3}, \end{aligned} \tag{11}$$

where $v_x = \cos\beta$, $v_y = \sin\beta$, and β characterizes the propagation direction of the light ray.

Equations (11) coincide with the standard equations of the light ray propagation in the RGT, and therefore the rotation of the solar system affects neither the magnitude of the deviation of light near the Sun nor the radio signal time lag.

We note that the variation of the eccentricity e and the major semiaxis a ($p = a(1 - e^2)$) of the orbit in the course of time is determined from the equations

$$\begin{aligned} \frac{de}{dt} &= \frac{1-e^2}{h}\Big(a \sin f A_\parallel + \frac{A_\perp}{e}\Big(\frac{ap}{r} - r\Big)\Big), \\ \frac{da}{dt} &= \frac{2a^2}{h}\Big(e \sin f A_\parallel + \frac{A_\perp p}{r}\Big). \end{aligned} \tag{12}$$

Substituting relations (8) into (12) and integrating over the total orbit we conclude that neither the galactic rotation nor the other terms in the equations of motion (6) make contribution to the integrated variation of the magnitudes of the eccentricity and the semiaxis:

$$\Delta e = 0, \qquad \Delta p = 0. \tag{13}$$

We stress that in the quasi-inertial coordinate system the equations of the motion of a test body differ from its equations of motion in the inertial coordinates, and therefore the absence of corrections additional to the anomalous ones in the above integrated effects in the quasi-inertial coordinate system does not at all mean the complete identity of physics in these different coordinate systems.

For example, the equation of the planar ($z' = 0$) motion of a test body has the following form (6) ($\mathbf{V}_c = (0,\ V_c,\ 0)$) in the coordinates $\mathbf{r} = (x',\ y',\ z')$ of the quasi-inertial system:

$$\begin{aligned} \ddot{\mathbf{r}} = &-\mathbf{r}m/r^3 + \mathbf{r}m/r^3[6\mathbf{V}_c^2 + 2(\mathbf{V}_c\dot{\mathbf{r}}) - \dot{\mathbf{r}}^2 + (3/2)(\mathbf{V}_c\mathbf{r})^2/r^2 + 4m/r] \\ &+ \dot{\mathbf{r}}m/r^3[(\mathbf{V}_c\mathbf{r}) + 4(\mathbf{r}\dot{\mathbf{r}})], \end{aligned}$$

where $\mathbf{V}_c$ is the linear velocity of galactic rotation.

In the polar coordinates ($x' = r\cos\varphi$, $y' = r\sin\varphi$) this equation is written as

$$\begin{aligned} \ddot{r} - r\dot{\varphi}^2 &= -m/r^2 + m/r^2[4m/r + 3\dot{r}^2 + 6V_c^2 - r^2\dot{\varphi}^2] \\ &\quad + \cos(\varphi)2mV_c\dot{\varphi}/r + \sin(\varphi)3m\dot{r}V_c/r^2 + \sin^2\varphi \cdot 3mV_c^2/(2r^2), \\ (r^2\dot{\varphi})^\cdot &= 4m\dot{r}\dot{\varphi} + \sin\varphi \cdot mV_c\dot{\varphi}. \end{aligned}$$

The circular orbit ($r = \text{const}$) is not a solution to the presented equations for $V_c \neq 0$. We now find the solution to these equations in the post-Newtonian approximation as a disturbance of the Newtonian circular orbit characterized by $r_n = a$, $\varphi_n = \omega t$, and $\omega_n = m^{1/2}/a^{3/2}$. We find

$$r = a + \cos(2z)aV_c^2/4,$$
$$\varphi = z - \sin z \cdot (m/a)^{1/2}V_c - \sin(2z)V_c^2/4,$$
$$z = \omega t, \quad \omega = \omega_n(1 - 3m/(2a) - 27V_c^2/8),$$

where ω is the post-Newtonian Kepler frequency.

This trajectory in the post-Newtonian approximation is an ellipse with eccentricity $e = V_c$. The body moves with a time-variable angular velocity and the equations of motion contain, along with ω, an additional harmonics 2ω. The enumerated effects are quite observable in their order of magnitude.

In conclusion we note that the calculations in the present paper were carried out using the program Reduce-3 for analytical calculations.

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31 May 1991

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