

PHASE SPACES OF NONZERO CURVATURE AND THE EVOLUTION OF PHYSICAL SYSTEMS

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It is proposed that phase spaces of nonzero curvature (non-Euclidean phase spaces) be used to describe physical phenomena. The singularities of these spaces show good correspondence to the invariant (equilibrium) states of physical systems. Pseudospherical phase spaces are investigated for which the Lobachevski geometry is valid. General principles are stated for the evolution of phenomena described, in particular, by the sin-Gordon, Korteweg-de Vries, Burgers, Liouville, and other equations.

1. INTRODUCTION

In the present paper a geometrical approach is announced for the interpretation of the dynamics of physical processes determined by nonlinear differential equations (the sin-Gordon, Korteweg-de Vries, Burgers, Liouville, and other equations.). The proposed method is based on the use of two-dimensional surfaces of nonzero curvature (which are related to the indicated equations) as two-dimensional phase spaces (surfaces) whose singularities are compatible with the invariant (equilibrium) states of physical systems. A detailed investigation of pseudospherical phase surfaces is presented. The general principles are stated for the development of phenomena of different nature, such as the propagation of shallow-water waves, the dynamics of Bloch's walls, or dislocations in crystals. It is noted that, on the whole, the laws governing the course of the physical processes under consideration are closely related to problems of applicability of the Lobachevski geometry in a Euclidean space.

2. THE GEOMETRY OF NONLINEAR EQUATIONS

Consider a partial differential equation

$$\mathcal{F}(u, u_x, u_t, \dots) = 0 \quad (1)$$

for an unknown function $u = u(x, t)$

Definition 1. Equation (1) is said to determine a metric of a surface of Gaussian curvature K in the three-dimensional Euclidean space E^3 if for any regular solution $u(x, t)$ to Eq. (1) it is always possible to write down a metric of a surface of the given curvature K :

$$ds_K^2 = E dx^2 + 2F dx dt + G dt^2, \quad (2)$$

where the coefficients E , F , and G depend on the solution u and its derivatives.

The subscript in ds_K^2 symbolizes the curvature of the metric.

Many equations of mathematical physics determine metrics of pseudospherical surfaces, i. e., surfaces of constant negative Gaussian curvature $K \equiv -1$. Presented below are examples of such equations and the corresponding metrics.

1) The sin-Gordon equation:

$$\begin{aligned} u_{xt} &= \sin u, \\ ds_{-1}^2 &= dx^2 + 2 \cos u dx dt + dt^2. \end{aligned} \quad (3)$$

2) The Korteweg-de Vries equation:

$$\begin{aligned} u_t + 6uu_x + u_{xxx} &= 0, \\ ds_{-1}^2 &= [(u-1)^2 + \eta^2] dx^2 \\ &\quad - 2[(u-1)(u_{xx} + \eta u_x + (\eta^2 - 2)u + 2u^2 - \eta^2) + \eta(\eta^3 + 2\eta u + 2u_x)] dx dt \\ &\quad + [(u_{xx} + \eta u_x + (\eta^2 - 2)u + 2u^2 - \eta^2)^2 + (\eta^3 + 2\eta u + 2u_x)^2] dt^2, \quad \eta = \text{const.} \end{aligned} \quad (4)$$

3) The elliptic Liouville equation:

$$\begin{aligned} \Delta u &= e^u, \\ ds_{-1}^2 &= (e^u/2)(dx^2 + dt^2). \end{aligned} \quad (5)$$

In [1, 2] some other examples of equations determining metrics of pseudospherical surfaces are also presented; moreover, in [2] conditions are found under which evolution equations admit of the above-mentioned geometrical interpretation.

The presented examples can be generalized to the case of an arbitrary Gaussian curvature $K(x, t)$. Namely, the equations generalizing (3) and (5) are written, respectively, as

$$u_{xt} = -K(x, t) \sin u, \quad (3')$$

$$\Delta u = -K(x, t) e^u. \quad (5')$$

Here the original equation (3) and its generalization (3') (and, similarly, (5) and (5')) determine metrics of qualitatively the same form.

Every equation (1) satisfying Definition 1 (and, in particular, Eqs. (3)-(5)) can be regarded as the Gauss equation for calculating the curvature of a surface with the corresponding metric (2).

Generally speaking, on the surfaces in E^3 there are singularities, such as irregular edges, cusps, etc. We now proceed to studying singularities of two-dimensional surfaces in E^3 .

3. SINGULARITIES OF TWO-DIMENSIONAL SURFACES IN E^3

In the space E^3 consider a surface S determined by the radius vector $\mathbf{r}(x, t)$, where x and t are intrinsic coordinates on the surface. The condition for the independence of the parameters x, t uniquely determining the function $\mathbf{r}(x, t)$ is written

$$[\mathbf{r}_x \times \mathbf{r}_t] \neq 0. \quad (6)$$

Condition (6) expresses the noncollinearity of the tangent vectors \mathbf{r}_x and \mathbf{r}_t to the coordinate lines on the surface. The possible points of the surface S where, instead of (6), the opposite condition

$$[\mathbf{r}_x \times \mathbf{r}_t] = 0 \quad (7)$$

holds, are called singular points (singularities) of the surface. The singularities also include the points of the surface where the function $\mathbf{r}(x, t)$ does not have at least one of its derivatives \mathbf{r}_x or \mathbf{r}_t . Singularities can also be of an extensive character, e.g., irregular edges that are envelopes of the points of contact of the coordinate lines.

In terms of metric (2), relations (7) are rewritten as

$$EG - F^2 = 0. \quad (8)$$

Consequently, the singularities of a surface S are uniquely determined by its metric.

Introduce the notation

$$\text{edge } \{S, \mathcal{F}(u, u_x, u_t, \dots) = 0\} \propto \{EG - F^2 = 0\}, \quad (9)$$

whose meaning is that the surface S with a metric described by Eq. (1) has a singularity determined by condition (8).

For the sin-Gordon equation (3) the singularities of the corresponding pseudospherical surfaces are determined by the conditions $u = n\pi$ (where n is an integer) [3, 4], and expression (9) takes the form

$$\text{edge}\{S, u_{xt} = \sin u\} \propto \{u = n\pi\} \quad (n \text{ is an integer}).$$

4. SURFACES WITH SINGULARITIES AS TWO-DIMENSIONAL PHASE SPACES

In this section we shall state the principles of the evolution of physical systems related to differential equations and their geometry. The suggested approach is based on the idea of using surfaces of nonzero curvature as two-dimensional phase spaces (surfaces), which are nonlinear analogs of classical phase spaces in theoretical physics. In contrast to Euclidean phase spaces (spaces of zero curvature) that are usually introduced in classical mechanics and statistical physics, phase spaces of nonzero curvature can have singularities; these singularities determine invariant (equilibrium) states of physical systems.

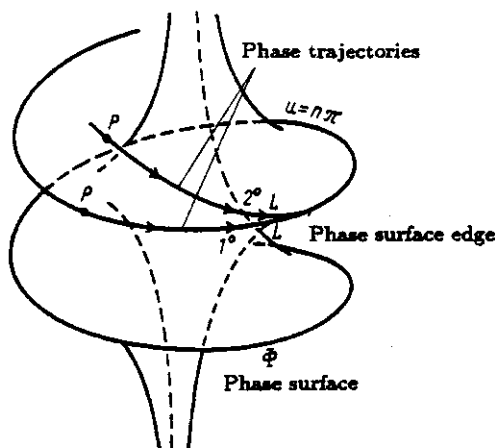


Fig. 1

Two qualitatively different versions of the behavior of a phase trajectory L on a phase surface Φ (the represented pseudospherical surface corresponds to a one-soliton solution to the sin-Gordon equation).

Consider a surface S with metric (2) determined by Eq. (1), which describes a physical process II. Assuming that, as a whole, over the entire surface S there is a regular network of lines (a regular network implies the regularity of all its lines as spatial curves) we take such a network as a coordinate system. Then the surface S can be interpreted as a phase surface Φ (in what follows every surface interpreted as a phase space will be denoted as Φ). A point P on the phase surface Φ will correspond to each state of the physical system (to definite values of the parameters x and t) and the motion of the point P along a line L (called a phase trajectory) on the phase surface Φ will correspond to the process itself (see Fig. 1). The phase trajectory L is a spatial curve in E^3 with radius vector r_L .

The known phase trajectories considered in different sections of theoretical physics (curves defining the development of natural processes) are regular curves. Probably this is due to the condition of correctness of the causality relationship between phenomena. Therefore we base the proposed consideration on the principle of *regularity of the phase trajectory*.

This principle determines the following two possible versions of the behavior of a phase trajectory on the surface Φ with singularities: 1) the phase trajectory (an unbounded line) lies entirely on the regular part

of the surface Φ and approaches its singularities asymptotically; and 2) the phase trajectory coincides with an irregular edge of the surface Φ (Fig. 1).

The condition of phase trajectory regularity precludes its intersecting the irregular edge of the phase surface.

The character of the geometrical behavior of a phase trajectory suggests the following principle of evolution of physical systems.

Principle 1. *Let a physical process Π be described by Eq. (1) determining metric (2) of the surface Φ (a surface with singularities). Then this process can develop in one of the two possible directions.*

1. *If at the initial time $t = t_0$ the condition*

$$(EG - F^2)|_{t=t_0} = 0$$

holds, then it will hold at all the subsequent instants of time:

$$\Pi : EG - F^2 = 0 \quad \forall t > t_0.$$

2. *In the case we have*

$$(EG - F^2)|_{t=t_0} \neq 0$$

at the time $t = t_0$, then, in the course of time, the physical system is stabilized in the following way:

$$\Pi : (EG - F^2) \rightarrow 0 \quad \text{for } t \rightarrow \infty.$$

The situation described by condition (8) will be called an invariant state of the system. It will be symbolized by the notation

$$\text{inv} \{ \Pi, \mathcal{F}(u, u_x, u_t, \dots) = 0 \}.$$

What has been said implies that

$$\text{inv} \{ \Pi, \mathcal{F}(u, u_x, u_t, \dots) = 0 \} \cap \text{edge} \{ \Phi, \mathcal{F}(u, u_x, u_t, \dots) = 0 \} \cap \{ EG - F^2 = 0 \}.$$

According to Principle 1, in the corresponding physical systems either an invariant state is realized or they approach it. In a sense, Point 1 of Principle 1 can be interpreted as a conservation law.

We shall elucidate Principle 1 using the example of propagation of shallow-water waves [5], i. e., we shall consider plane waves on a fluid surface whose maximum perturbation amplitude a is small compared to the depth h ($\varepsilon = a/h \ll 1$) whereas the perturbation length l (wavelength) is much greater than h ($\delta = h/l \ll 1$). The investigation of this problem by methods of perturbation theory (here ε and δ play the role of small parameters) leads to the Korteweg-de Vries equation (4) where the unknown function $u(x, t)$ has the meaning of the wave amplitude. This system has the following invariant state:

$$\text{inv} \{ \Pi, u_t + 6uu_x + u_{xxx} = 0 \} \cap \{ u_x = 0 \}.$$

Consequently, in accordance with Point 2 of Principle 1, the process of propagation of shallow-water waves arrives at the state $u_x = 0$ (or $u = \text{const}$), which corresponds to the well-known damping of these waves. Point 1 of Principle 1 represents the unperturbed state of the fluid surface.

We now apply Principle 1 to the sin-Gordon equation (3).

Principle 2. *Let the quantity u^* determining a physical process satisfy the sin-Gordon equation (3). Then*

1. *the values $u^* = n\pi$ (where n is an integer) are invariants of this physical process;*

2. if at an instant of time we have $u^* \neq n\pi$ (n is an integer), then, in the course of time, the function u^* tends asymptotically and monotonically to a multiple of π so that its variation during the whole process is less than π .

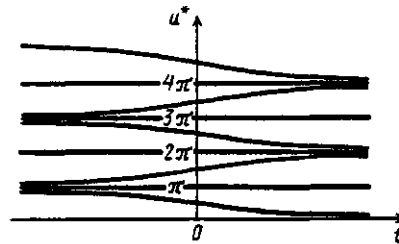


Fig. 2

Qualitative structure of the observed quantity u^* in phenomena described by the sin-Gordon equation. The values of u^* are obtained from the analytical solution $u(x, t)$ of the sin-Gordon equation by separating out its values on the phase trajectory.

According to Point 1 of Principle 2, in a real physical process the function u^* can assume the value $n\pi$ only in the case when u^* is identically equal to $n\pi$ during the whole process (the state of $n\pi$ -invariance).

We now give some examples of phenomena obeying Principle 2 (note that the geometrical and physical meaning of this principle was analyzed in [6, 7]).

The state of $n\pi$ -invariance corresponds to the following phenomena: 1) self-induced transparency manifesting itself when ultrashort pulses propagate in two-level resonant media (passage of the energy of pulses, whose area is a multiple of π , through a medium without losses) [8, 9]; 2) equilibrium positions of atoms in crystal lattices [10]; 3) vacuum states of mesons [11]; 4) topologically invariant states of elementary particles [12]; and 5) stable states of the magnetization vector orientation in an external magnetic field in ferromagnetics [13]. Actually, the state of $n\pi$ -invariance represents an equilibrium state of a system.

Point 2 of Principle 2 describes, for instance, the following phenomena: 1) variation of the area of an ultrashort pulse [8, 9]; 2) instability of intermediate positions of atoms in crystals and the discrete character of dislocations [10]; 3) relaxation of perturbed states of elementary particles to a vacuum state [11]; 4) decay of the current in the Josephson contact [14]; and 5) rotation of the magnetization vector in the Bloch "180°-wall" [13].

The meaning of Principle 2 is demonstrated qualitatively in Fig. 2.

It is important to note that the quantity u^* observed in experiment differs from the analytical solution $u(x, t)$ to the sin-Gordon equation. Namely, u^* is obtained from the solution $u(x, t)$ by separating out its values on the phase trajectory:

$$u^* = u(x, t)|_{(x,t) \in L},$$

and the variation of u^* over the entire observation time is less than π because of the regularity of the phase trajectory L .

Thus, in experiment one can observe the quantity u^* and not the analytical solution to the sin-Gordon equation as such (Fig. 2). Generally, the questions of experimental observability in phenomena obeying Principle 2 are closely related to the problem of isometric embeddings of parts of the Lobachevski plane in the three-dimensional space E^3 [3, 7], which means that, in a sense, such phenomena can be studied in terms of the Lobachevski geometry.

Table 1 gives invariant states for different equations of mathematical physics.

Table 1

Differential Equations and Invariant States of Physical Systems

Equation	Invariant state of physical system	Equation	Invariant state of physical system
1. The sin-Gordon equation $u_{xt} = \sin u$	$u = n\pi$, n is integer	5. The hyperbolic Liouville equation $u_{xt} = e^u$	$u_x = 0$
2. The Korteweg-de Vries equation $u_t + 6uu_x + u_{xxx} = 0$	$u_x = 0$	6. The elliptic Liouville equation $\Delta u = e^u$	$u \rightarrow -\infty$
3. The modified Korteweg-de Vries equation $u_t + 6u^2u_x + u_{xxx} = 0$	$u_x = 0$	7. The hyperbolic sinh-Gordon equation $u_{xt} = \sinh u$	$u_x = 0$
4. The Burgers equation $u_t + uu_x + u_{xx} = 0$	$u_x = 0$	8. The elliptic sinh-Gordon equation $\Delta u = \sinh u$	$u = 0$

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