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# THE PROPERTIES OF THE SPECTRUM OF A CLASS OF NATURAL VIBRATIONS OF OPEN CYLINDRICAL RESONATORS

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**The properties of the spectrum of natural vibrations are considered for open cylindrical resonators whose cross section is formed by a half-plane with finite irregularity of the boundary. This property allows a correct setting of the partial Reichardt–Sveshnikov conditions at infinity. The generally formulated problem reduces to analyzing the Fredholm operator function of the frequency spectrum parameter defined on the Riemann surface of the analytical continuation of the fundamental solution. The discrete character of the spectrum and the absence of finite limit points are proved. Regions free of spectrum points are characterized.**

Let $\Pi_{ab}^{\infty}$ be a class of unbounded finite two-dimensional regions $\Omega$ that form the cross section of the family of resonators under consideration. In these regions, setting of boundary value problems for the Helmholtz equation with generalized partial conditions at infinity of the Reichardt–Sveshnikov type is justified [1, 2]. Earlier, starting with [1, 3], similar conditions and their modifications [4, 5] were applied either to cylindrical regions (regular waveguides) with local discontinuities (see, e. g., [6, 7]), or to infinite regions with compact [2, 8] or noncompact periodic [5, 9] boundaries. Extension of the partial conditions to problems with local inhomogeneities in a half-plane or half-space with a locally nonuniform boundary and a description of a class of regions similar to $\Pi_{ab}^{\infty}$ are given in [7] (see also [9–11]).

Put $x = (x_1, x_2) \in R^2$. Noncompact portions of the boundaries of regions $\Omega$ from class $\Pi_{ab}^{\infty}$ are formed by two rays $\{x | x_2 = 0, x_1 \leqslant a, x_1 \geqslant b\}$. Let $U_d(0) = \{x| \; |x| < d\}$, and let there exist the smallest positive $R_0$ such that $\Omega \backslash \overline{U}_{R_0}(0) = \{x | x_2 > 0; |x| > R_0\}; a, b \in U_{R_0}(0)$. The boundary of $\Omega$ in $U_{R_0}(0)$ satisfies the cone condition [12] and is a finitely connected set. Examples of regions of this type are shown in Fig. 1.

Cylindrical structures with cross sections formed by regions $\Omega$ are usually called open slit (microstrip, mirror, etc.) lines or resonators [7].

This class of structures includes both various planar microstrip, mirror, and slit resonators and resonators formed by screens with arbitrary finite inhomogeneities such as rectangular grooves. These resonators can also contain a finite number of open and closed infinitely thin screens situated above the inhomogeneities and a finite number of dielectric inclusions.

Let us formulate the mathematical problem of natural vibrations of the $E$- and $H$-types.

In $\Omega$, consider the equation

$$\Delta u(x) + \lambda \varepsilon(x) u(x) = 0, \tag{1}$$

$$x = (x_1, x_2) \in \Omega \subset R^2, \tag{1'}$$

where $\varepsilon(x)$ is the permittivity of the medium represented by a positive piecewise constant function, which outside $U_{R_0}$ takes on a fixed constant value $\varepsilon_1$, and $\lambda$ is a complex spectral parameter. The $u$ function coincides with the longitudinal component of the electric (magnetic) field.

Let $A$ be a set of points of the $\varepsilon(x)$ function discontinuity lines. Each region of $A$ satisfies the conjugation

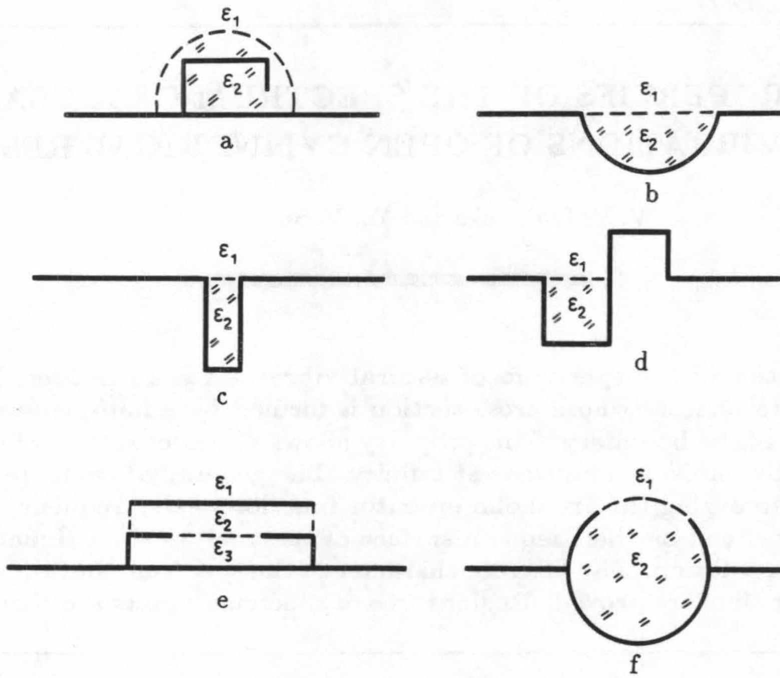


**Fig. 1**

conditions

$$[u]|_A = \left[\frac{\partial u}{\partial n}\right]\Big|_A = 0, \tag{2}$$

$$[u]|_A = \left[\frac{1}{\varepsilon}\frac{\partial u}{\partial n}\right]\Big|_A = 0. \tag{2'}$$

The boundary conditions met on $\partial\Omega$ are

$$u|_{\partial\Omega} = 0, \tag{3}$$

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0, \tag{3'}$$

where $\mathbf{n}$ is a normal to $\partial\Omega$.

In the neighborhoods of the cusps of $\partial\Omega$ (sharp edges), "on-edge" conditions are set. These conditions correspond to the requirement that the electromagnetic field energy be finite in any neighborhood $S$ of such a point, if this neighborhood belongs to $\Omega$:

$$\int\!\!\int_S (|\nabla u|^2 + |u|^2)\, dx < \infty. \tag{4), (4'}$$

At infinity, radiation conditions in the Reichardt–Sveshnikov form [1] are imposed, that is, for $r = \sqrt{x_1^2 + x_2^2} > R_0$, the solution in the upper half-plane is representable as the series

$$u(r, \varphi) = \sum_{n=1}^{\infty} C_n H_n^{(1)}(\sqrt{\varepsilon_1 \lambda}\, r) \sin n\varphi, \tag{5}$$

$$u(r, \varphi) = \sum_{n=0}^{\infty} C_n H_n^{(1)}(\sqrt{\varepsilon_1 \lambda}\, r) \cos n\varphi, \tag{5'}$$

which can be termwise differentiated with respect to $r$ and $\varphi$, and which uniformly converges in $\varphi \in [0, \pi]$ at any fixed $r$. Here $H_n^{(1)}(z)$ is the Hankel function of the first kind.

Let us define the functional class $M = \{u | u \in C^2(\Omega \backslash A) \bigcap C^1(\overline{\Omega}_i \backslash S_\delta),\ i = 1, \dots, n\}$, where $S_\delta$ is the union of the $\delta$-neighborhoods of the cusps of the boundary of $\partial\Omega$, and $\Omega_i = \{x | \varepsilon(x) = \varepsilon_i = \text{const}\}$.

**Definition 1.** The problem of determining the spectrum of two-dimensional natural vibrations (1) through (5) ((1′) through (5′)) involves the search for nontrivial solutions (eigenfunctions) of Eq. (1) ((1′)) from class $M$ satisfying conditions (2) through (5) ((2′) through (5′)). The values of the spectral parameter $\lambda$ for which such solutions exist are called eigennumbers of problem (1) through (5) ((1′) through (5′)).

The eigenfunctions of these problems describe the natural vibrations corresponding to the natural frequency $\omega$ for a cylindrical resonator with the cross section $\Omega$. The frequency $\omega$ and the parameter $\lambda$ are related as $\omega^2 = \lambda c^2$, where $c$ is the velocity of light in the vacuum.

First consider problem (1′) through (5′). Next we shall make necessary comments and additional remarks concerning problem (1) through (5).

Let $\lambda$ be a fixed value of the spectral parameter of problem (1′)–(5′), and $R > R_0$ be a number which, for the given $\lambda$, is not a root of the equations $H_n^{(1)}(\sqrt{\varepsilon_1 \lambda}\, r) = 0$, where $n \in N$. Note that, as has been shown in [13], this condition holds with all $R > R_0$ for $\lambda$ such that $4k\pi \leqslant \operatorname{Arg}\lambda \leqslant 2(2k+1)\pi$, where $k \in Z$.

Next, let the functions $u$ and $v$ belong to the class $M$ and $u$ be an eigenfunction of problem (1′)–(5′). At a fixed $r > R$,

$$u(r, \varphi) = \sum_{n=0}^{\infty} C_n H_n^{(1)}(\sqrt{\varepsilon_1 \lambda}\, r) \cos n\varphi \tag{6}$$

is the Fourier series in the $\{\cos n\varphi,\ n = 0, 1, \dots\}$ set of functions. The $u$ function is doubly continuously differentiable at $r > R_0$. Therefore the requirement that the left-hand side values be equal to the right-hand side ones on the $C_R = \{x|\ |x| = R\} \bigcap \{x | x_2 > 0\}$ semicircle yields the coefficients in the form

$$\begin{aligned} C_n &= \frac{2}{\pi} \int_0^\pi u(R, \varphi) \cos n\varphi \, d\varphi / H_n^{(1)}(\sqrt{\varepsilon_1 \lambda}\, R), \qquad n \geqslant 1, \\ C_0 &= \frac{1}{\pi} \int_0^\pi u(R, \varphi)\, d\varphi / H_0^{(1)}(\sqrt{\varepsilon_1 \lambda}\, R). \end{aligned} \tag{7}$$

Clearly, the expression for the normal derivative of $u$ on the $C_R$ curve takes the form

$$\begin{aligned} b_\lambda^R = \frac{\partial u}{\partial r}\bigg|_R &= \frac{1}{\pi} \int_0^\pi u(R, \varphi)\, d\varphi \frac{\partial H_0^{(1)}(\sqrt{\varepsilon_1 \lambda}\, R)/\partial r}{H_0^{(1)}(\sqrt{\varepsilon_1 \lambda}\, R)} \\ &+ \sum_{n=1}^{\infty} \frac{2}{\pi} \int_0^\pi u(R, \psi) \cos n\psi \, d\psi \frac{\partial H_n^{(1)}(\sqrt{\varepsilon_1 \lambda}\, R)/\partial r}{H_n^{(1)}(\sqrt{\varepsilon_1 \lambda}\, R)} \cos n\varphi. \end{aligned} \tag{8}$$

After dividing Eq. (1′) by $\varepsilon(x) > 0$, multiplying the quotient by $\overline{v}$, and integrating the result by parts in the $\Omega_R = \Omega \bigcap U_R$ region taking Eq. (2′) into account, we obtain the integral equality

$$\iint_{\Omega_R} \Big( \frac{1}{\varepsilon} \nabla u \nabla \overline{v} - \lambda u \overline{v} \Big)\, dx - \int_{C_R} \frac{1}{\varepsilon} b_\lambda^R(u) \overline{v}\, dl = 0. \tag{9}$$

Note that the boundary of the $\Omega_R$ region satisfies the cone condition.

**Definition 2.** The nontrivial element $u \in W_2^1(\Omega_R)$ satisfying equality (9) for any $v \in W_2^1(\Omega_R)$ function and representable by series (5′) with coefficients (7) in $\Omega \backslash \overline{\Omega}_R$ will be called a generalized eigenfunction (GEF) of problem (1′) through (5′).

It can be shown that every classical eigenfunction of problem (1′)–(5′) is generalized.

Let us prove the following assertion: the definition of GEFs is independent of $R$. Indeed, let $u_{R_1}$ and $u_{R_2}$ be the GEFs at $R = R_1$ and $R = R_2$, respectively, and let $R_2 > R_1 > R_0$. It can be shown that $u_{R_1}$ and

$u_{R_2}$ fit the definition at $R = R_2$ and $R = R_1$, respectively. Because a GEF can be represented by series (5′) at $r > R_0$, the $u_{R_i}$, $i = 1, 2$, functions satisfy the equality

$$\iint\limits_{\Omega_{R_1}^{R_2}} \Big( \frac{1}{\varepsilon} \nabla u_{R_i} \nabla \bar{v} - \lambda u_{R_i} \bar{v} \Big)\, dx - \int\limits_{C_{R_1}} \frac{1}{\varepsilon_1} \frac{\partial u_{R_i}}{\partial r} \bar{v}\, dl + \int\limits_{C_{R_2}} \frac{1}{\varepsilon_1} \frac{\partial u_{R_i}}{\partial r} \bar{v}\, dl = 0, \tag{10}$$

where $\Omega_{R_1}^{R_2} = \{x | R_1 < |x| < R_2\}$. The validity of this assertion can be checked by applying theorems on internal smoothness of generalized solutions [14] and the standard technique of separation of variables. Adding Eqs. (10) and (9) at $R = R_1$, $i = 1$, and subtracting Eq. (10) from (9) at $R = R_2$, $i = 2$, completes the proof.

Let $D_\lambda^R$ be a simply connected region on the Riemann surface of the $H_n^{(1)}(\sqrt{\varepsilon_1 \lambda} R)$ function whose closure does not contain zero. Taking into account the asymptotic behavior of the Hankel functions and the equality

$$H_n^{(1)\prime}(z) = i \Big( \frac{2}{\pi z} \Big)^{1/2} e^{i(z - \pi n/2 - \pi/4)} \Big( 1 + O\Big(\frac{1}{|z|}\Big) \Big) \quad \text{for } |z| \to \infty, \tag{11}$$

we can find sufficiently large $R > R_0$ such that, first, $H_n^{(1)}(\sqrt{\varepsilon_1 \lambda} R)$ will have no roots in $D_\lambda^R$ at any $n \in Z^+$, and, secondly, for all $\lambda$ from $D_\lambda^R$

$$\big| (\partial H_n^{(1)}(\sqrt{\varepsilon_1 \lambda} R)/\partial r)/H_n^{(1)}(\sqrt{\varepsilon_1 \lambda} R) \big| \leqslant |\sqrt{\varepsilon_1 \lambda}|(1 + C_1) = C_2(\lambda), \tag{12}$$

where $C_1$ is some small positive constant.

Using the statements on the compactness of embedding (12) makes it possible to show that for every $\lambda \in D_\lambda^R$, equality (9) is equivalent to the operator equation

$$u - (\lambda + 1) A^H u - B^H(\lambda) u = F \tag{13}$$

in $W_2^1(\Omega_R)$, where $A^H$ is a completely continuous operator in $W_2^1(\Omega_R)$. Similar operator equations were obtained in [15] for problems on propagation of waves in waveguides with irregular filling.

Consider the properties of the $B^H(\lambda)$ operator function defined as

$$\iint\limits_{\Omega_R} \Big( \frac{1}{\varepsilon} \nabla B^H u \nabla \bar{v} + B^H u \bar{v} \Big)\, dx = \int\limits_{C_R} \frac{1}{\varepsilon} b_\lambda^R(u) \bar{v}\, dl. \tag{14}$$

Using representation (8), estimate (12), the Parseval equality, and the properties of traces of functions from $W_2^1(\Omega_R)$ makes it possible to prove the boundedness of sesquilinear form (14) in $W_2^1(\Omega_R)$. This in turn implies complete continuity of $B^H(\lambda)$ at each $\lambda$ value from $D_\lambda^R$ (as a consequence of the compactness of embedding of the traces of a bounded set of functions from $W_2^1(\Omega_R)$ into $L_2(C_R)$).

Further, by virtue of the selection of $R$ and on the basis of the Weierstrass theorem, we can, using the results obtained in [16], conclude that the operator function $B^H(\lambda)$ is analytic in $D_\lambda^R$.

This implies the validity of the following

**Lemma 1.** Problem (1′) through (5′) of determining the spectrum of natural vibrations in the generalized formulation is equivalent in $D_\lambda^R$ to the problem of characteristic numbers

$$K^H(\lambda) u = 0 \tag{15}$$

for the analytic Fredholm operator function

$$K^H(\lambda) = 1 - (\lambda + 1) A^H - B^H(\lambda)$$

that acts in $W_2^1(\Omega_R)$.

The equivalence is to be understood as follows: a nontrivial solution to Eq. (15) at a fixed $\lambda \in D_\lambda^R$ extended to $\Omega$ by Eq. (5′) with coefficients (7) is the GEF of problem (1′)–(5′) that satisfies Eq. (15) in $W_2^1(\Omega_R)$.

Now consider problem (1) through (5). The $C_n$ coefficients in Eq. (5) are given by the formulae

$$C_n = \frac{2}{\pi}\int_0^\pi u(R,\,\varphi)\sin n\varphi\,d\varphi/H_n^{(1)}(\sqrt{\varepsilon_1\lambda}\,R), \qquad n\in N, \tag{16}$$

and $b_\lambda^H(u)$ can be represented as

$$b_\lambda^R(u) = \sum_{n=1}^{\infty}\frac{2}{\pi}\int_0^\pi u(R,\,\psi)\sin n\psi\,d\psi\frac{\partial H_n^{(1)}(\sqrt{\varepsilon_1\lambda}\,R)/\partial r}{H_n^{(1)}(\sqrt{\varepsilon_1\lambda}\,R)}\sin n\varphi.$$

The classical eigenfunctions of problem (1)–(5) satisfy the integral equality

$$\iint_{\Omega_R}(\nabla u\nabla\overline{v} - \lambda\varepsilon(x)u\overline{v})\,dx - \int_{C_R} b_\lambda^H(u)\overline{v}\,dl = 0 \tag{17}$$

for any function $v$ that is continuously differentiable in $\Omega_R$ and satisfies the condition $\operatorname{supp} v\subset\Omega$.

**Definition 3.** A nonzero $u\in\widetilde{W}_2^1(\Omega_R)$ function satisfying equality (17) for any $v\in\widetilde{W}_2^1(\Omega_R)$ and representable in $\Omega\backslash\Omega_R$ by series (5) with coefficients (16) is called the GEF of problem (1) through (5).

In this definition, $\widetilde{W}_2^1(\Omega_R)$ denotes the subset of functions from $W_2^1(\Omega_R)$ with a zero trace on $\partial\Omega$. Virtually in the same way we arrive at

**Lemma 2.** Problem (1) through (5) in the generalized formulation is equivalent in $D_\lambda^R$ to the problem of characteristic numbers

$$K^E(\lambda)w = 0$$

for the analytic operator function $K^E(\lambda) = 1-(\lambda+1)A^E - B^E(\lambda)$, which is a Fredholm operator in $\widetilde{W}_2^1(\Omega_R)$.

Let the function $\varepsilon(x)$ satisfy the following condition: for any point in the region $\Omega$ there exists a smooth curve such that one of its ends coincides with the selected point and the other belongs to the set $\Omega\bigcap\{x|\,|x|>R_0\}$. Each intersection of this curve with the set $A\subset\Omega$, on which $\varepsilon(x)$ experiences discontinuity, has a neighborhood where $A$ is an infinitely smooth curve. Let $E$ be the set of positive piecewise constant functions satisfying this condition and taking on a fixed constant value at $|x|>R_0$.

**Lemma 3.** Let $\varepsilon(x)\in E$. Then the set $\{\lambda|2k\pi\leqslant\operatorname{Arg}\lambda\leqslant 2(2k+1)\pi,\ k\in Z\}$ does not contain points of the spectrum of problem (1) through (5) ((1′) through (5′)).

The proof will be given for problem (1)–(5) in the generalized formulation. A similar approach applies to problem (1′)–(5′).

For simplicity, suppose that $\varepsilon(x)$ takes on two values, $\varepsilon_1$ and $\varepsilon_2$, that is, $\Omega=\Omega_1\bigcup\Omega_2\bigcup A$, where $\Omega_1=\{x|\varepsilon(x)=\varepsilon_1\}$ is an unbounded subdomain of $\Omega$.

Choose $R>R_0$. At all $v\in\widetilde{W}_2^1(\Omega_R)$, the GEF of problem (1)–(5) meets equality (17) and satisfies Eq. (1) in the $\Omega\backslash\Omega_R$ region. Multiplying Eq. (1) by $\overline{u}$ and integrating the product by parts in the $\Omega_R^{R_1}$ ($R_1>R$) region, we obtain

$$\iint_{\Omega_R^{R_1}}(\,|\nabla u|^2 - \lambda\varepsilon_1|u|^2\,)\,dx + \int_{C_R} b_\lambda^R(u)\overline{u}\,dl - \int_{C_R}\frac{\partial u}{\partial r}\overline{u}\,dl = 0. \tag{18}$$

With $v=u$ in Eq. (17), adding Eqs. (17) and (18) yields

$$\iint_{\Omega_{R_1}}(\,|\nabla u|^2 - \lambda\varepsilon(x)|u|^2\,)\,dx = \int_{C_{R_1}}\frac{\partial u}{\partial r}\overline{u}\,dl. \tag{19}$$

Suppose that $\lambda\in\{\lambda|4k\pi<\operatorname{Arg}\lambda<2(2k+1)\pi\}$ and let $R_1$ in Eq. (19) tend to infinity. Then, by virtue of Eq. (5) and the asymptotic representation of Hankel functions, we arrive at the conclusion that $u=0$

almost everywhere in $\Omega$, which proves the lemma for the specified $\lambda$ values. Next, let $\operatorname{Arg}\lambda = 2k\pi$, which implies that $\sqrt{\lambda}$ is real. Clearly, for all $v \in \widetilde{W}_2^1(\Omega_R)$, we then have

$$\iint\limits_{\Omega_R} (\nabla\overline{u}\nabla v - \lambda\varepsilon(x)\overline{u}v)\,dx = \int\limits_{C_R} \overline{b_\lambda^R(u)}v\,dl. \tag{20}$$

With $v = u$ in Eqs. (17) and (20), subtracting the latter equation from the former termwise yields

$$\int\limits_{C_R} (b_\lambda^R(u)\overline{u} - \overline{b_\lambda^R(u)}u)\,dl = 0.$$

Further, again using the assumption that $\sqrt{\lambda}$ is real, we arrive at the chain of equalities

$$0 = \int\limits_{C_R} (b_\lambda^R(u)\overline{u} - \overline{b_\lambda^R(u)}u)\,dl = \sum_{n=1}^{\infty}\int_0^{\pi} \sin n\varphi|C_n|^2 W\big[H_n^{(1)}(\sqrt{\varepsilon_1\lambda}\,R),$$

$$H_n^{(1)}(\sqrt{\varepsilon_1\lambda}\,R)\big]R\,d\varphi + \sum_{n=1}^{\infty}\sum_{\substack{s=1\\ s\neq n-s}}^{\infty}\int_0^{\pi} \sin s\varphi\sin(n-s)\varphi(\overline{C}_{n-s}C_sH_{n-s}^{(1)}(\sqrt{\varepsilon_1\lambda}\,R)$$

$$\times H_n^{(1)'}(\sqrt{\varepsilon_1\lambda}\,R) - C_{n-s}\overline{C}_sH_{n-s}^{(1)}(\sqrt{\varepsilon_1\lambda}\,R)H_s^{(1)'}(\sqrt{\varepsilon_1\lambda}\,R))R\,d\varphi$$

$$= \frac{\pi}{2}\sum_{n=1}^{\infty}|C_n|^2,$$

where $C = RW\big[H_n^{(1)}(\sqrt{\varepsilon_1\lambda}\,R),\ H_n^{(1)}(\sqrt{\varepsilon_1\lambda}\,R)\big]$. It follows that all $C_n$ are zero. Because the definition of the GEF of problem (1)–(5) in the generalized formulation is independent of $R$, $u = 0$ for $r > R_0$. Further, by virtue of the theorems of internal smoothness, the embedding theorems [14], the corresponding integral representations, and the independence of $R$ of the definition of GEFs, $u$ is an analytic function in the $\Omega_1$ and $\Omega_2$ regions, which immediately yields $u = 0$ in $\Omega_1$. As $u \in W_2^2(G)$ for any $G : \overline{G} \subset \Omega_R$ region and is a piecewise analytic function in $\Omega$, it follows from the corresponding theorems on the boundary values (distributions) of analytic functions [17] on an infinitely smooth region of $G$ that $u = 0$ in $\Omega$, which completes the proof of the lemma.

Thus, the independence of $R$ of the definition of GEFs, Lemmas 1 through 3, the results obtained in [18], and the arbitrariness of selecting $D_\lambda^R$ lead us to the principal result of this work in the form of the following theorem.

**Theorem.** The spectrum of problem (1) through (5) ((1′) through (5′)) is discrete. This spectrum comprises eigennumbers of finite multiplicities, does not belong to the $\{\lambda|\operatorname{Im}\sqrt{\lambda} \geqslant 0\}$ set, and has no finite limit points in $C\backslash\{0\}$.

This theorem establishes the fundamental property of a discrete character of the spectrum of natural vibrations of the considered class of open cylindrical resonators with noncompact boundaries and dielectric inclusions. This result complements the known propositions of a discrete character of the spectra of other classes of open resonators with noncompact boundaries (see, e.g., [5, 10]).

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