Moscow University Physics Bulletin
Vol. 53, No. 1, pp. 68–69, 1998

Vestnik Moskovskogo Universiteta. Fizika
UDC 523.11

BRIEF COMMUNICATIONS

THEORETICAL AND MATHEMATICAL PHYSICS

CASIMIR ENERGY ON A MULTISTRING SPACE

Yu. V. Grats

The energy density of vacuum fluctuations on a multistring space is known to have nonintegrable singularities, hence the total vacuum energy per unit string length diverges. The Casimir effect study is shown to lead to necessary additional regularization corresponding to renormalization of bare values of angle deficits for strings.

Some aspects of total vacuum fluctuation energy calculation on nontrivial topology spaces are of interest from various points of view.

In what follows we will deal with an ultrastatic space which is a direct product of the two-dimensional Minkowski space and a two-dimensional locally flat space with a set of conic singularities. It is known [1] that the corresponding metric is a solution of Einstein's equation with the energy-momentum tensor of the system of parallel cosmic strings in its right-hand side. We shall call this the multistring space.

We define the total vacuum oscillation energy as the limiting value of the field energy as the inverse temperature β tends to infinity,

$$E_{\text{vac}} = \lim_{\beta\to\infty} E_\beta. \tag{1}$$

To calculate E_β we use the relation

$$E_\beta = \frac{\partial}{\partial\beta} W_\beta, \tag{2}$$

where

$$e^{-W_\beta} = \int D[\phi] e^{-S_E[\phi]}. \tag{3}$$

In (3), $S_E[\phi]$ is the Euclidean action and integration is carried out over all field configurations that are determined in the Euclidean sector of the space-time under consideration and are periodic with respect to imaginary time (with period β).

In the case of a real scalar field we have

$$S_E[\phi] = \frac{1}{2}\int\limits_0^\beta d\tau \int \sqrt{g} d^{D-1}x \phi(x) L_D \phi(x), \tag{4}$$

where $L_D = -\Delta_D + \xi R + m^2$, Δ_D is the Laplace–Beltrami operator, D is the space dimension.

It is known that the effective action can be written in terms of the generalized ζ-function,

$$W_\beta[\phi] = \frac{1}{2}\ln\det\left(\frac{L_D}{\lambda^2}\right) = -\frac{1}{2}\zeta'_\beta\left(0\middle|\frac{L_D}{\lambda^2}\right). \tag{5}$$

In this expression, an arbitrary parameter λ has dimension of mass. It is introduced for reasons of dimensions.

In an ultrastatic space with Euclidean sector $[0,\beta] \times M_{D-1}$, direct calculation gives [2]

$$\zeta_\beta(s|L_D) = \frac{\beta}{\sqrt{4\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta_\infty\left(s - \frac{1}{2}\middle|L_{D-1}\right) + \frac{2\beta}{\sqrt{4\pi}\Gamma(s)} \sum_i \sum_{n=1}^{\infty} \left(\frac{2E_i}{n\beta}\right)^{(1/2)-s} K_{(1/2)-s}(n\beta E_i), \quad (6)$$

where E_i^2 are eigenvalues of the L_{D-1} operator, and $K_{(1/2)-s}$ is the MacDonald function.

The Euclidean sector of the multistring space under consideration is a particular case of the space with structure $[0,\beta] \times R \times M_2$, where M_2 is a locally flat surface with conical singularities.

Using (6) makes it possible to show that for the space $[0,\beta] \times R \times M_2$ the following relations hold:

$$\frac{\partial}{\partial\beta}\zeta_\beta(0|L_4) = -\frac{\zeta_\infty(-1|L_2)}{4\pi}\int dz,$$

$$\lim_{\beta\to\infty} \frac{\partial}{\partial\beta}\zeta'_\beta(0|L_4) = -\frac{\zeta'_\infty(-1|L_2) + \zeta_\infty(-1|L_2)}{4\pi}\int dz. \quad (7)$$

Here z is a coordinate in R. (In the multistring space it is the coordinate along the string.) Equations (2), (5), and (7) give the following formal relation:

$$E_{\text{vac}} = \frac{1}{4\pi}[\zeta'_\infty(-1|L_2) + (1 + \ln\lambda^2)\zeta_\infty(-1|L_2)]\int dz. \quad (8)$$

If M_2 is a multiconic surface, the local ζ-function, $\zeta(-1,*,*|L_2)$, has nonintegrable singularities at the cones vertexes, and the integral

$$\mathrm{Tr}\,\zeta_\infty(-1,*,*|L_2),$$

and the vacuum energy (8) diverge. Hence additional regularization is required. The idea is that the Casimir energy proper is not a physical magnitude. By the Casimir energy is meant the part of the renormalized total energy that emerges due to an external gravitational field. The space with conic singularities is not obviously Ricci-flat. In the case of a multistring 4-D space the bare energy of classical matter has the structure of (8),

$$E_m = \sum_{i=1}^{N} \mu_i \int dz. \quad (9)$$

Combining (9) with the vacuum fluctuation energy expression yields

$$\frac{E_{\text{tot}}}{\int dz} = \sum_{i=1}^{N} \mu_i + \frac{1}{8\pi}\left[\zeta'_\infty(-1|L_2) + (1 + \ln\lambda^2)\zeta_\infty(-1|L_2)\right]. \quad (10)$$

Thus, basing upon the generalized ζ-function method, we showed that the singularities in (8) can be included in the renormalized total angle deficit on M_2. The Casimir energy proper is given by the finite part of (8). For a single string the corresponding result was obtained by another technique [3].

The generalized ζ-function of the Laplace operator on a multiconic two-dimensional surface has not been studied yet. However, the finite part of $\zeta(-1|L_2)$ can be expected to depend on the singularities relative arrangement. This means that the Casimir interaction exists between the cones vertexes. This assumption is justified by the results obtained earlier by the perturbation theory procedures [4].

The work was supported by the Russian Foundation for Basic Research (Grant 97-02-18003).

REFERENCES

1. P.S. Letelier, *Class. Quantum Grav.*, vol. 4, p. L75, 1987.
2. K. Kirsten and E. Elizalde, *E-print*, hep-th/9707083.
3. D.V. Fursaev, *Preprint JINR*, E2-93-291, 1993.
4. D.V. Galtsov, Yu.V. Grats, and A.B. Lavrentyev, *Yad. Fiz.*, vol. 58, p. 516, 1995.

19 September 1997

Department of Theoretical Physics