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DIFFERENTIAL-GEOMETRIC CRITERION FOR KINEMATIC INTEGRABILITY OF EQUATIONS WITH OPERATORS FROM $su(1,1)$ AND $su(2)$

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A necessary and sufficient condition for kinematic integrability of equations with operators belonging to the Lie algebras $su(1,1)$ and $su(2)$ is proved. The suggested criterion is related to differential-geometric consideration of the corresponding classes of kinematically integrable equations and is based on the G-representation of partial differential equations.

THE NOTION OF G-CLASS

Let

$$ds^2 = g_{ij}dx^i dx^j, \quad (x^1 = x, \quad x^2 = t)$$

be a metric on a smooth two-dimensional manifold. We assume that its coefficients can be represented in the form $g_{ij} = g_{ij}(u, u_x, u_t, u_{xx}, \ldots; x, t) \equiv g_{ij}[u]$, where $u(x,t)$ is an unknown function. If the Gaussian curvature $K(x,t)$ is given, then the Gauss equation (see, e.g., [1]) is a generally nonlinear differential equation with respect to the function $u(x,t)$,

$$f[u(x,t)] = 0. \tag{1}$$

In this case, the corresponding metric tensor $g_{ij} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is called a G-representation of equation (1). Equations admitting G-representations are said to belong to the G-class. The G-representations (G-class) corresponding to the Gaussian curvature $K \equiv -1$ are referred to as the Λ^2-representations (Λ^2-class).

The notion of Λ^2-class, which has comparatively recently been introduced by Poznyak and Popov [2, 3], is a connecting link between nonlinear equations and the differential geometry of two-dimensional smooth manifolds.

In what follows, we denote the G-representations corresponding to constant nonzero Gaussian curvatures as $G\{f[u] = 0\}$, $K \equiv \text{const} \neq 0$, representations.

REPRESENTATION OF ZERO CURVATURE

Consider the problem

$$\begin{cases} \psi_x = U\psi, \\ \psi_t = V\psi, \end{cases}$$

where U and V are the (2×2)-matrix operators and $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is the two-dimensional vector-function.

The solvability condition for this system is obtained by cross differentiation and has the form

$$U_t - V_x + [U, V] = 0. \tag{2}$$

Equation (2) is called a zero curvature representation equation. If the operators U and V analytically depend on a certain parameter ξ, then the equations representable in form (2) are called kinematically integrable equations [4].

CONSTRUCTING THE ZERO CURVATURE REPRESENTATION OPERATORS GIVEN THE $G\{F[U] = 0\}$, $K \equiv \text{const} \neq 0$, REPRESENTATION

Given a $G\{f[u] = 0\}$, $K \equiv \text{const} \neq 0$, representation of a certain differential equation

$$f[u] = 0. \tag{3}$$

In [5], the theorem stated below is proved. It relates the kinematic integrability [4] of equations to their membership in the G-class and allows us to construct the spectral-evolution operators U and V by the G-representation ($K \equiv \text{const} \neq 0$) of equation (3) so that the zero curvature representation equation for U and V coincides with the corresponding Gauss equation.

Theorem 1. *Given a G-representation of equation* (3), *i. e., the corresponding metric tensor* $g_{ij}[u]$, *and the Gaussian curvature* $K \equiv \text{const} \neq 0$.

Then

1. *The operators*

$$U = \begin{bmatrix} \frac{i}{2}\widetilde{a} & \frac{1}{2}\sqrt{-K}\sqrt{E}\exp(i\theta^+) \\ \frac{1}{2}\sqrt{-K}\sqrt{E}\exp(-i\theta^+) & -\frac{i}{2}\widetilde{a} \end{bmatrix}$$

and

$$V = \begin{bmatrix} \frac{i}{2}\widetilde{b} & \frac{1}{2}\sqrt{-K}\sqrt{G}\exp(i\theta^-) \\ \frac{1}{2}\sqrt{-K}\sqrt{G}\exp(-i\theta^-) & -\frac{i}{2}\widetilde{b} \end{bmatrix},$$

where

$$\theta^{\pm} = \pm\frac{1}{2}\arccos\left(\frac{F}{\sqrt{EG}}\right), \quad \widetilde{a} = \frac{1}{2\sqrt{w}}\left[\frac{FG_x}{2G} - \frac{FE_x}{2E} + F_x - E_t\right],$$

$$\widetilde{b} = \frac{1}{2\sqrt{w}}\left[\frac{FG_t}{2G} - \frac{FE_t}{2E} - F_t + E_x\right], \quad w = EG - F^2,$$

satisfy the zero curvature representation equation

$$U_t - V_x + [U, V] = 0. \tag{4}$$

2. *Equation* (4) *coincides with the Gauss equation* $f[u] = 0$ *corresponding to the given* $G\{f[u] = 0\}$, $K \equiv \text{const} \neq 0$, *representation.*

3. *The operators U and V are unique to a gauge transformation.*

Theorem 1 implies that the set of kinematically integrable equations includes the set of all equations that admit $G\{f[u] = 0\}$, $K \equiv \text{const} \neq 0$, representations.

CONSTRUCTING G-REPRESENTATIONS GIVEN ZERO CURVATURE OPERATORS

Consider the problem inverse to that discussed above. Namely, let us construct the inverse representation for a metric tensor for the given zero curvature representation operators.

Theorem 2. 1. *Given operators $U = U[u(x,t)]$ and $V = V[u(x,t)]$ of the form*

$$U = \begin{pmatrix} U^{11} & U^{12} \\ U^{21} & -U^{11} \end{pmatrix}, \quad V = \begin{pmatrix} V^{11} & V^{21} \\ V^{21} & -V^{11} \end{pmatrix}$$

satisfying the zero curvature representation equation

$$U_t - V_x + [U, V] = 0 \tag{5}$$

for a certain equation $f[u(x,t)] = 0$.

2. *Let one of the following conditions be fulfilled:*

(a) $U, V \in su(2)$,

(b) $U, V \in su(1,1)$.

Then

1°. *There exists a G-representation of the equation* $f[u(x,t)] = 0$ *for a Gaussian curvature* $K \equiv \text{const} \neq 0$, *where* $K > 0$ *in case (a) and* $K < 0$ *in case (b).*

2°. *The G-representation under consideration is expressed by the metric*

$$ds^2 = -\frac{4U^{12}U^{21}}{K}dx^2 - 2\frac{2(U^{12}V^{21} + U^{21}V^{12})}{K}dx\,dt - \frac{4V^{12}V^{21}}{K}dt^2. \tag{6}$$

A DIFFERENTIAL-GEOMETRIC CRITERION FOR KINEMATIC INTEGRABILITY

Let us denote the set of kinematically integrable equations with matrix operators $U, V \in su(1,1)$ ($U, V \in su(2)$) by $P_{su(1,1)}$ ($P_{su(2)}$). The following inclusions are obvious corollaries to Theorem 2:

$$\begin{gathered} P_{su(1,1)} \subseteq G\{K \equiv \text{const} < 0\}, \\ P_{su(2)} \subseteq G\{K \equiv \text{const} > 0\}. \end{gathered} \tag{7}$$

Theorem 1 gives the inverse inclusions

$$\begin{gathered} P_{su(1,1)} \supseteq G\{K \equiv \text{const} < 0\}, \\ P_{su(2)} \supseteq G\{K \equiv \text{const} > 0\}. \end{gathered} \tag{8}$$

Assertions (7) and (8) imply the equivalence of $P_{su(1,1)}$ to $G\{K \equiv \text{const} < 0\}$ and of $P_{su(2)}$ to $G\{K \equiv \text{const} >)\}$.

These equivalences of the corresponding classes make it possible to state the following criterion for kinematic integrability of equations.

Theorem 3. *An equation* $f[u(x,t)] = 0$ *belongs to the class of kinematically integrable equations with matrix operators* $U, V \in su(1,1)$ ($U, V \in su(2)$) *if and only if the equation* $f[u(x,t)] = 0$ *belongs to the* $G\{K \equiv \text{const} < 0\}$*-class* ($G\{K \equiv \text{const} > 0\}$*-class*).

Theorem 3 is a differential-geometric criterion, because it gives necessary and sufficient conditions for kinematic integrability of equations.

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REFERENCES

1. B.A. Dubrovin, S.P. Novikov, and A.T. Fomenko, *Modern Geometry: Methods and Applications* (in Russian), Nauka, Moscow, 1986.
2. E.G. Poznyak and A.G. Popov, *Dokl. Ross. Akad. Nauk*, vol. 332, no. 4, p. 418, 1993.
3. E.G. Poznyak and A.G. Popov, in: *Advances in Science and Technology, Ser. Geometry-2* (in Russian), p. 5, Moscow, 1994.
4. L.A. Takhtadzhyan and L.D. Fadeev, *Hamiltonian Approach to the Theory of Solitons* (in Russian), Nauka, Moscow, 1986.
5. S.A. Zadadaev, *Candidate's of Science (Phys.-Math.) Thesis*, Moscow, 1999.

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