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FERROMAGNETIC STATE OF THE $SU(2)$-VACUUM

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The conditions for the ferromagnetic state of the chromomagnetic vacuum to exist are considered. It is shown that tachyonic modes can condense into a spatially uniform state if the length of the chromomagnetic field is limited. The problem of the phase transition between the ferromagnetic and superconducting states is considered.

1. Gauge configurations, satisfying the Yang–Mills equations and providing local minima of the gauge action, are of great importance for the explanation of various non-perturbative phenomena in quantum theory of non-Abelian fields. One of these solutions is a constant chromomagnetic field [1]. It was shown in [2] that, for certain densities, the state of quarks interacting with the gauge field is energetically more advantageous, when they occupy Landau levels in a nonzero chromomagnetic field ("ferromagnetic" state) than when they are inside the Fermi sphere in the absence of a field. However, a constant chromomagnetic field, though being a solution of the Yang–Mills equations, does not correspond to a minimum of the action due to tachyonic modes present in the excitations about this field. This problem has been studied in a number of papers ([2, 3] are among the recent studies). In this paper, attempt is made to study the phase transition between the ferromagnetic state of a vacuum and the phase without a chromomagnetic field.

2. We consider a pure gauge theory with the $SU(2)$ group in a $(3+1)$-dimensional space-time. The Lagrangian $L = -1/4 F^a_{\mu\nu} F^a_{\mu\nu}$ can be rewritten in the form that demonstrates explicitly the interaction between a charged vector field and an "electromagnetic" field,

$$L = -\frac{1}{4} f^2_{\mu\nu} - \frac{1}{2} |(D_\mu \Phi_\nu - D_\nu \Phi_\mu)|^2 - ie f_{\mu\nu} \Phi^+_\mu \Phi_\nu + \frac{e^2}{4} (\Phi^+_\mu \Phi_\nu - \Phi^+_\nu \Phi_\mu)^2, \tag{1}$$

where the following notation is introduced: $A_\mu = A^3_\mu$ is a neutral ("electromagnetic") field; $\Phi_\mu = \frac{1}{\sqrt{2}}(A^1_\mu + iA^2_\mu)$ is a charged field, $D_\mu = \partial_\mu + ieA_\mu$.

In our problem, the "electromagnetic" field is chosen to be a constant chromomagnetic field directed along the z-axis, the gauge is antisymmetric, and a charged field may be chosen vanishing,

$$\begin{aligned} A_\mu &= (A_0, \mathbf{A}) = (0, (0, x_1 B, 0)), \\ \Phi_\mu &= \Phi^+_\mu = 0. \end{aligned} \tag{2}$$

This set (A, Φ) solves the field equations. However, when nonvanishing solutions of the field equations are found for the charged field Φ in an external chromomagnetic field A_μ, they can be written in the form $\Phi_\mu = \mathrm{e}^{-iEx_0 + ik_3x_3 + ik_2x_2} f_{n\mu}(x_1 - k_2/eB)$. The corresponding energy spectrum looks like (see, e. g., [4])

$$E^2 = k_3^2 + 2eB(n + \frac{1}{2}) \pm 2eB, \tag{3}$$

where $n = 0, 1, 2, \ldots$. It is clear that the energy becomes imaginary, when the spin orientation corresponds to the minus sign, the quantum number $n = 0$ and $k_3^2 < eB$ (tachyonic modes). This demonstrates that this configuration is unstable with respect to shifts along tachyonic modes described by these energies [5].

The authors of [2, 3] suggest a method which allows one to reach the uniform quantum Hall state by going down along the unstable modes. The direction of descent is chosen to suit the requirement that the final solution must be spatially uniform. Only modes with $k_3 = 0$ are chosen so that there is no dependence on x_3. To eliminate the dependence on x_2, the averaging over all possible values of k_2 is required. It is evident that in this case the most "rigid" modes are excited. The solution is sought in the form of functions independent of x_3,

$$\Phi_\mu(x) = \Phi_\mu(x_0, x_1, x_2)\sqrt{L_3}, \tag{4}$$

where L_3 is the length of chromomagnetic field in the direction of the x_3-axis. Taking into account the fact that the tensor of the neutral field has only one nonvanishing component f_{12}, we may describe the unstable mode by the following Lagrangian for the scalar field φ:

$$L_{\text{eff}} = -\frac{1}{2}|D_\mu\varphi|^2 + 2eB\varphi^2 - \frac{e^2}{2L_3}\varphi^4. \tag{5}$$

Lagrangian (5) is similar to the Higgs Lagrangian if we consider the term describing the interaction with the constant chromomagnetic field as a negative squared mass. The only and rather important difference is that the interaction with the chromomagnetic field enters also the covariant derivatives. As a result, there are no spatially uniform solutions of the field equations with this Lagrangian. In order to make possible their presence, a method similar to that used in [6] for fermions was applied in [2]. Its main idea is that we go over to another effective $(2+1)$-dimensional model with a Chern–Simons term describing new bosons φ_a,

$$L_{\text{eff}} = -\frac{1}{2}|(i\partial_\mu - eA_\mu + a_\mu)\varphi_a|^2 + 2eB\varphi_a^2 - \frac{e^2}{2l}\varphi_a^4 + \frac{\epsilon_{\mu\nu\lambda}}{4\alpha}a_\mu\partial_\nu a_\lambda. \tag{6}$$

As is known [7], the Chern–Simons term changes the statistics of particles, "attaching" to them an extra magnetic flux, and making it, generally speaking, fractional depending on the values of parameter α. The statistics is conserved only at $\alpha = 2\pi k$, where k is an integer. In [2, 6], it is demonstrated that in this case the old model (5) and the new model (6) are dynamically equivalent. The field φ_a describes new bosons that condense to the spatially uniform solution, $\varphi_a = v = \text{const}$. This solution arises when the Chern–Simons gauge field a_μ totally compensates for the constant chromomagnetic field A_μ in the covariant derivatives and the model becomes a pure Higgs model, $a_i = eA_i$.

We can safely guarantee that the found solution is a true vacuum, i.e., it is also stable against perturbations relatively nonuniform along the chromomagnetic field direction, only when there are no tachyonic modes with the momentum $k_3 \neq 0$.

Relation (3) implies that all the modes with $n = 0$, $s = -1/2$ and $k_3^2 < eB$ are unstable. In order to exclude modes with $k_3 \neq 0$, it is sufficient to make the extension of the field along the x_3-axis finite. In this case, imposing on the charged field the periodicity conditions along this axis, $A_\mu^{1,2}(0) = A_\mu^{1,2}(L_3)$, the discretization of the corresponding momentum follows: $k_3 = 2\pi n_3/L_3$. In order to avoid the unstable modes with $k_3 \neq 0$, we should require that the squared energy of the lowest modes with $n_3 = \pm 1$ be positive, and this imposes a physical restriction on the maximal possible extention of the chromomagnetic field along the x_3-axis,

$$\left(\frac{2\pi}{L_3^{\max}}\right)^2 = eB. \tag{7}$$

The chromomagnetic field by itself does not break the color neutrality. However, it is broken by the presence of the charged field condensate. Since only white states are admissible, we come to the conclusion that the aforementioned state is forbidden in the pure gauge theory. According to [2], the quark field should be introduced as a supplier of the color charge, and this allows one to restore the color neutrality.

3. As we have seen, the gauge configuration consisting of the constant chromomagnetic field and the charged uniform condensate can exist only in the presence of fermions. We will consider massless fermions in the fundamental representation. They are described by the Dirac equation

$$i\gamma_\mu D_\mu\Psi = 0, \tag{8}$$

where $D_\mu = \partial_\mu + ieA_\mu$. In every gauge configuration, the Dirac equation leads to a particular energy spectrum. In the ground state, fermions with finite density will occupy a certain number of lower levels. In this way, their total energy, which is the sum of all occupied energy levels, will depend on the external gauge field. The restrictions are as follows: the space–time is $(3+1)$-dimensional, though the extension of the chromomagnetic field in the x_3 direction is limited by the interval $(0, L_3)$, where $L_3 = L_3^{\max} = 2\pi/\sqrt{eB}$, and the system should be neutral colored. Our aim is to investigate the system for the case of constant chromomagnetic fields considering the restrictions imposed on the boson sector, and to answer the question about the phase transition between the color ferromagnetic state ($B \neq 0$) and color superconducting ($B = 0$) state. In other words, would it be energetically advantageous to "switch off" the chromomagnetic field without changing L_3?

The "Fermi energy" is defined, for the cases both with and without an external field, from the equation

$$N = \sum_\lambda \Theta(E_F - E_\lambda), \tag{9}$$

where λ is the set of quantum numbers for the Dirac wave functions both with and without the field. The total energy is defined as

$$E_{\text{tot}} = \sum_\lambda \Theta(E_F(N) - E_\lambda)E_\lambda. \tag{10}$$

Solutions of the free Dirac equation have the form of plane waves and are determined by the quantum numbers: $\lambda = c, s, \epsilon, k_1, k_2, k_3$, where $c, s, \epsilon = \pm 1$ are the color, the spin projection, and the sign of the energy, $E_\lambda^2 = \mathbf{k}^2$. Considering the periodicity conditions leads to the restriction on the values of the momenta: $k_i = 2\pi n_i / L_i$, where n_i are integers.

When $B \neq 0$, the quantum numbers and the spectrum are also well known: $\lambda = c, s, \epsilon, n, k_2, k_3$,

$$E_\lambda^2 = \frac{e}{2}B(2n + 1 - s) + k_3^2,$$

where the effective charge is considered to be equal to $e' = e/2$. The periodicity condition again determines possible values of the momenta, and the form of the wave function limits the maximal possible value of k_2, and this determines the degree of degeneracy with this quantum number.

Taking into account (7), we can write equations for both cases in a unique form:

$$\begin{aligned} N(E_F, L_3) &= \frac{4\pi L_1 L_2}{L_3^2} f(\frac{E_F L_3}{2\pi}), \\ E_{\text{tot}}(E_F, L_3) &= \frac{8\pi^2 L_1 L_2}{L_3^3} g(\frac{E_F L_3}{2\pi}), \end{aligned} \tag{11}$$

where f, g are dimensionless functions of a dimensionless variable, defined according to the relations

$$\begin{aligned} f^{B=0}(x) &= \sum_{n_3=-\infty}^{\infty} 2\Theta(x^2 - n_3^2)(x^2 - n_3^2), \\ g^{B=0}(x) &= \sum_{n_3=-\infty}^{\infty} 4/3\Theta(x^2 - n_3^2)(x^3 - n_3^3) \end{aligned} \tag{12}$$

in zero chromomagnetic field, and

$$\begin{aligned} f^{B\neq 0}(x) &= \sum_{n_3=-\infty}^{\infty} \sum_{n=0}^{\infty} (2 - \delta_{n0})\Theta(x^2 - n - n_3^2), \\ g^{B\neq 0}(x) &= \sum_{n_3=-\infty}^{\infty} \sum_{n=0}^{\infty} (2 - \delta_{n0})\Theta(x^2 - n - n_3^2)\sqrt{n + n_3^2} \end{aligned} \tag{13}$$

in the presence of a chromomagnetic field.

With the notation $h(x) = g(f^{-1}(2\pi^2 x))$, the total energy can be written as

$$E_{\text{tot}} = 2e^2 B^2 V h\left(\frac{\rho}{(eB)^{3/2}}\right), \tag{14}$$

where $\rho = N/V = N/(L_1 L_2 L_3)$ is the density of fermions. The energy depends on B in the case of free fermions because in fact it depends on finite L_3 which can be written in terms of B.

We consider the question about the energy gain due to the presence of the chromomagnetic field. It is obvious that due to its generation the fermion energy will change by the amount $-2(eB)^2 V\Delta h(\rho/(eB)^{3/2})$, where $\Delta h = h^{B=0} - h^B$. We shall add to it the energy of the chromomagnetic field itself. Furthermore, if $B = 0$ and supposing there exists a mechanism of effective attraction, the energy could be gained due to the effect of color superconductivity [8] (the influence of a chromomagnetic field on the color superconductivity and chiral symmetry breaking effects is discussed in [9]): fermions lying near the Fermi surface create Cooper pairs gaining the energy $V2\pi\rho^{2/3}\Delta^2$, where Δ is the coupling energy of a Cooper pair. Thus, the total energy change due to the presence of the chromomagnetic field is written as

$$\Delta E = VB^2\left(-2e^2\Delta h\left(\frac{\rho}{(eB)^{3/2}}\right) + \frac{1}{8\pi} + \frac{2\pi\rho^{2/3}\Delta^{2/3}}{B^2}\right). \tag{15}$$

Function $\Delta h(x)$ was calculated numerically and is depicted in Fig 1. It is clear that it is positive and undergoes frequent oscillations related to discreteness of the Landau levels. Whether a ferromagnetic phase is advantageous depends on how the negative term in (15), that defines the energy gain due to fermions changing their places from zero field levels to Landau levels, can compete with the positive terms related to energy losses for the chromomagnetic field production and destroying the superconductivity mechanism.

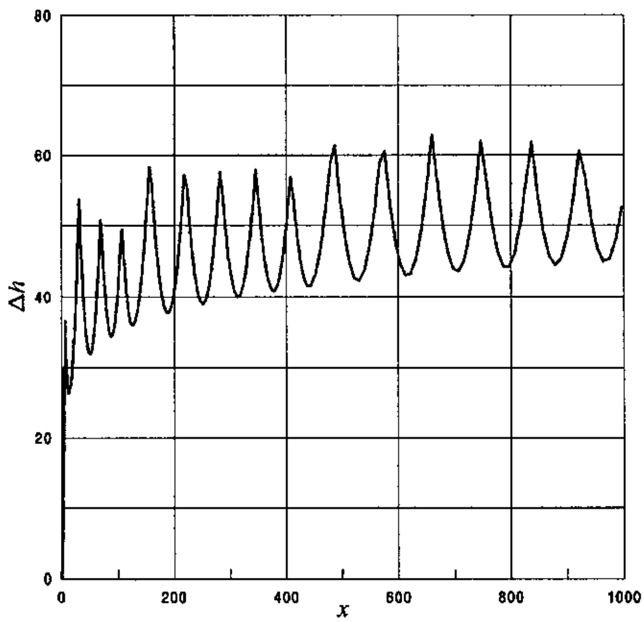

Fig. 1

Function $\Delta h(x)$.

The competition of the first two terms is evidently reduced to the question of how large the coupling constant is. For instance, when the characteristic dimension L_3 of the system tends to zero, the running coupling constant is also going to zero and production of a chromomagnetic field becomes disadvantageous. Hence, this indicates the lower bound for L_3 and an upper bound for B.

The relation between the first and the third terms is determined by the value of the fermion density. The condition that the ferromagnetic state is energetically advantageous imposes an upper bound on the fermion density.

In their subsequent studies, the authors plan to consider a more realistic model in order to make particular estimates of the abovementioned facts, and to be able to make a more definite answer to the question whether a ferromagnetic phase in the QCD vacuum exists.

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