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HELMHOLTZ DIFFERENTIAL EQUATION WITH AN OBLIQUE DERIVATIVE AND DIRICHLET CONDITION SPECIFIED ON OPPOSITE SIDES OF CUTS IN A PLANE

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Existence and uniqueness theorems for the solution of a boundary-value problem for a nonwave Helmholtz differential equation, where the Dirichlet condition and condition with an oblique derivative are specified on the opposite sides of cuts in a plane, are proved. An integral representation for the solution is obtained.

Let us consider on the plane $x = (x_1, x_2) \in \mathbb{R}^2$ a set of simple open curves $\Gamma_1, \dots, \Gamma_N$ of class $C^{2,\lambda}$, $\lambda \in (0, 1]$, having no common points, the ends included. We will refer to this set as the contour Γ. Let the contour Γ be parametrized, the parametrization parameter being the arc abscissa (length of arc) s: $\Gamma_n = \{x\colon x = x(s) = (x_1(s), x_2(s)),\ s \in [a_n, b_n]\}$, $n = 1, \dots, N$. We select parametrization such that for different n the line segments $[a_n, b_n]$ on the axis $0s$ have no common points, including the ends. The vector of the tangent line to Γ at the point $x(s)$ that indicates the direction of increase of the parameter s will be designated as $\tau_x = \{\cos\alpha(s), \sin\alpha(s)\}$, and the vector of the normal that coincides with the vector of the tangent line upon rotation through an angle of $\pi/2$ counterclockwise, as $\mathbf{n}_x = \{\sin\alpha(s), -\cos\alpha(s)\}$. With the above parametrization, $x_1'(s) = \cos\alpha(s)$, $x_2'(s) = \sin\alpha(s)$. The set of the segments of the axis $0s$ that correspond to the contour Γ, will also be designated as Γ. We will say that the function $\mathcal{F}(s)$ defined on Γ belongs to the Banach space $C^r_\varkappa(\Gamma)$, $r \in (0, 1]$, $\varkappa \in [0, 1)$, provided that $\mathcal{F}_0(s) = \mathcal{F}(s)\prod_{n=1}^N |[(s - a_n)(s - b_n)]^\varkappa| \in C^{0,r}(\Gamma)$. The norm in the space $C^r_\varkappa(\Gamma)$ is defined by the formula $\|\mathcal{F}(s)\|_{C^r_\varkappa(\Gamma)} = \|\mathcal{F}_0(s)\|_{C^{0,r}(\Gamma)}$. We assume that the plane $\mathbb{R}^2$ is cut along the contour Γ. We denote by Γ^+ that side of the contour Γ, which remains on the left as the parameter s grows and by Γ^- the opposite side. We use the symbol X to designate the set of the points of the plane that consists of the ends of the contour Γ: $X = \bigcup_{n=1}^N (x(a_n) \cup x(b_n))$.

We will say that the function $u(x) = u(x_1, x_2)$ belongs to the smoothness class $\mathcal{G}$ if (1) $u(x) \in C^0(\overline{\mathbb{R}^2 \setminus \Gamma})$, i. e., $u(x)$ is continuous outside of the cuts of Γ, is continuously extendable to the cuts of Γ from the left and right at all points, and also continuously extendable to the ends of the cuts of Γ; (2) $u_{x_1}, u_{x_2} \in C^0(\overline{\mathbb{R}^2 \setminus \Gamma} \setminus X)$, where X is the set of the ends of the contour Γ; and (3) at the ends of the cuts of Γ the functions u_{x_1}, u_{x_2} can have integrable singularities, i. e., as $x \to x(d) \in X$, the following inequality holds true for certain constants $\delta > -1$, $A > 0$:

$$|u_{x_j}(x)| \le A|x - x(d)|^\delta, \quad j = 1, 2, \tag{1}$$

where $d = a_n$ or $d = b_n$, $n = 1, \dots, N$.

Let us formulate a mixed problem for a Helmholtz differential equation outside of a system of cuts in a plane.

Problem $\mathcal{U}$. To find: The real function $u(x)$ from the class $\mathcal{G}$ that satisfies in $\mathbb{R}^2 \setminus \Gamma$ in the classical sense the Helmholtz equation

$$\Delta u - k^2 u = 0, \quad k = \operatorname{Re} k > 0, \tag{2}$$

the boundary conditions

$$u(x)|_{\Gamma^+} = f^+(s), \tag{3}$$

$$\left(\frac{\partial u}{\partial \mathbf{n}_x} + \beta \frac{\partial u}{\partial \tau_x}\right)\bigg|_{\Gamma^-} = f^-(s), \tag{4}$$

and the conditions at infinity

$$|u(x)| = o\left(|x|^{-1/2}\right), \quad |\nabla u| = o\left(|x|^{-1/2}\right), \quad |x| = \left(x_1^2 + x_2^2\right)^{1/2} \to \infty. \tag{5}$$

We assume that $f^+(s), f^-(s)$ are specified real functions and that β is a specified real constant.

An analog of Problem $\mathcal{U}$ for the Laplace differential equation was studied in [1]. Then, by $\int_\Gamma \ldots ds$ we will mean $\sum_{n=1}^{N} \int_{a_n}^{b_n} \ldots ds$. Using the method of energetic identities, we can prove the following theorem.

Theorem 1. *If $\Gamma \in C^{2,\lambda}$, $\lambda \in (0,1]$, then Problem $\mathcal{U}$ has no more than one solution.*

We will construct the solution of Problem $\mathcal{U}$ assuming that $f^+(s) \in C^{1,\lambda}(\Gamma)$, $f^-(s) \in C^{0,\lambda}(\Gamma)$. We note that the Hoelder coefficient λ is taken to be the same in defining the smoothness of both the contour Γ and the functions $f^+(s)$, $f^-(s)$. If the value of this coefficient differs between these two cases, the least one should be used. Instead of boundary condition (3) we write the equivalent condition

$$\frac{\partial u}{\partial \tau_x}\bigg|_{\Gamma^+} = f'^+(s), \quad f'^+(s) \equiv \frac{df^+(s)}{ds} \in C^{0,\lambda}(\Gamma), \tag{6}$$

$$u(x(a_n)) = f^+(a_n), \quad n = 1, \ldots, N. \tag{7}$$

We denote by $K_0(z)$ the zero-order Macdonald function [2] that is a singular solution of equation (2).

We will construct the solution of Problem $\mathcal{U}$ by means of the potential's theory for equation (2). We seek the solution of Problem $\mathcal{U}$ in the form

$$u[\mu, \nu](x) = V[\mu](x) + T[\nu](x), \tag{8}$$

where $V[\mu](x) = \frac{1}{2\pi}\int_\Gamma \mu(\sigma) K_0(k|x - y(\sigma)|)\, d\sigma$ is the potential of a simple layer for equation (2) and $T[\nu](x) = \frac{1}{2\pi}\int_\Gamma \nu(\sigma) U(x,\sigma)\, d\sigma$ is the angular potential for equation (2). The angular potential for the Helmholtz equation was studied in [3]. The densities $\mu(s)$, $\nu(s)$ will be sought for in the Banach space $C^r_\varkappa(\Gamma)$, $r \in (0,1]$, $\varkappa \in [0,1)$. The angular potential's kernel $U(x,\sigma)$ is defined on each arc Γ_n $(n = 1, \ldots, N)$ by the formula

$$U(x,\sigma) = \int_{a_n}^{\sigma} \frac{\partial}{\partial \mathbf{n}_y} K_0(k|x - y(\xi)|)\, d\xi, \quad \sigma \in [a_n, b_n],$$

where $y = y(\xi) = (y_1(\xi), y_2(\xi))$, $|x - y(\xi)| = \sqrt{(x_1 - y_1(\xi))^2 + (x_2 - y_2(\xi))^2}$. In what follows, we will assume that the angular potential density satisfies the additional conditions [3, 4]

$$\int_{a_n}^{b_n} \nu(\sigma)\, d\sigma = 0, \quad n = 1, \ldots, N. \tag{9}$$

Integrating $T[\nu](x)$ by parts and using (9), we express the angular potential in terms of the double-layer potential

$$T[\nu](x) = -\frac{1}{2\pi} \int_\Gamma \rho(\sigma) \frac{\partial}{\partial \mathbf{n}_y} K_0(k|x - y(\sigma)|)\, d\sigma,$$

where $\rho(\sigma) = \int_{a_n}^{\sigma} \nu(\xi)\, d\xi$, $\sigma \in [a_n, b_n]$, $n = 1, \ldots, N$. Obviously the potentials $T[\nu](x)$ and $V[\mu](x)$ satisfy both equation (2) outside of Γ and the conditions at infinity (5).

Analogs of the potentials $T[\nu](x)$ and $V[\mu](x)$ that contained in their kernels the zero-order Hankel function of first kind, $H_0^{(1)}(z)$, in place of the Macdonald function were studied in [3]. Inasmuch as $H_0^{(1)}(z) = -\frac{2i}{\pi}K_0(-iz)$, the theory of the potentials $T[\nu](x)$ and $V[\mu](x)$ is in essence contained in [3], and so we will make use of the results of that work. It has been demonstrated in [3] that if $\mu(s), \nu(s) \in C^r_\varkappa(\Gamma)$, $r \in (0,1]$, $\varkappa \in [0,1)$ and conditions (9) are satisfied, function (8) belongs to the class $\mathcal{G}$ and meets all the conditions of Problem $\mathcal{U}$, except for boundary conditions (3), (4). Specifically, inequality (1) is satisfied at $\delta = -\varkappa$ if $\varkappa \in (0,1)$.

To meet the boundary conditions, we substitute function (8) into expressions (4), (6), using the limiting formulas for the angular potential from [3], and obtain the following integral equations for the densities $\mu(s)$, $\nu(s)$ on Γ:

$$\begin{gathered}\frac{1}{2\pi}\int\limits_\Gamma \frac{\partial}{\partial s}(\mu(\sigma)K_0(k|x(s)-y(\sigma)|)+\nu(\sigma)U(x(s),\sigma))\,d\sigma + \frac{\nu(s)}{2} = f'^{+}(s),\\ -\frac{\mu(s)+\beta\nu(s)}{2} + \frac{1}{2\pi}\int\limits_\Gamma \frac{\partial}{\partial \mathbf{n}_x}(\mu(\sigma)K_0(k|x(s)-y(\sigma)|)+\nu(\sigma)U(x(s),\sigma))\,d\sigma \\ +\beta\frac{1}{2\pi}\int\limits_\Gamma \frac{\partial}{\partial s}(\mu(\sigma)K_0(k|x(s)-y(\sigma)|)+\nu(\sigma)U(x(s),\sigma))\,d\sigma = f^{-}(s).\end{gathered} \tag{10}$$

Substituting function (8) into conditions (7), we obtain additional equations for $\nu(s)$, $\mu(s)$,

$$V[\mu](x(a_n)) + T[\nu](x(a_n)) = f^{+}(a_n), \quad n = 1,\dots,N. \tag{11}$$

The following theorem issues from the above reasoning.

Theorem 2. *Let $\Gamma \in C^{2,\lambda}$, $f^{+}(s) \in C^{1,\lambda}(\Gamma)$, $f^{-}(s) \in C^{0,\lambda}(\Gamma)$, $\lambda \in (0,1]$. If the system of equations (9)–(11) has a solution $\nu(s)$, $\mu(s)$ such that $\nu(s), \mu(s) \in C^r_\varkappa(\Gamma)$, $r \in (0,1]$, $\varkappa \in [0,1)$, the solution of Problem $\mathcal{U}$ then exists and is given by formula* (8).

Equations (10) can be written in the form ($s \in \Gamma$)

$$\begin{gathered}\nu(s) + \frac{1}{\pi}\int\limits_\Gamma \mu(\sigma)\frac{d\sigma}{\sigma - s} + \int\limits_\Gamma \nu(\sigma)v_1(s,\sigma)\,d\sigma + \int\limits_\Gamma \mu(\sigma)v_2(s,\sigma)\,d\sigma = 2f'^{+}(s),\\ -\mu(s) - \beta\nu(s) - \frac{1}{\pi}\int\limits_\Gamma (\nu(\sigma)-\beta\mu(\sigma))\frac{d\sigma}{\sigma - s} + \int\limits_\Gamma \mu(\sigma)v_3(s,\sigma)\,d\sigma - \int\limits_\Gamma \nu(\sigma)v_4(s,\sigma)\,d\sigma = 2f^{-}(s),\end{gathered} \tag{12}$$

where $v_1(s,\sigma) = \frac{1}{\pi}\frac{\partial}{\partial s}U(x(s),\sigma)$, $v_2(s,\sigma) = \frac{1}{\pi}\frac{\partial}{\partial s}K_0(k|x(s)-y(\sigma)|) - \frac{1}{\pi(\sigma - s)}$, $v_3(s,\sigma) = \frac{\beta}{\pi}(\frac{\partial}{\partial s}K_0(k|x(s) - y(\sigma)|) - \frac{1}{\sigma - s}) + \frac{1}{\pi}\frac{\partial}{\partial \mathbf{n}_x}K_0(k|x(s)-y(\sigma)|)$, $v_4(s,\sigma) = -\frac{1}{\pi}\frac{\partial}{\partial \mathbf{n}_x}U(x(s),\sigma) - \frac{1}{\pi(\sigma - s)} - \frac{\beta}{\pi}\frac{\partial}{\partial s}U(x(s),\sigma)$. Since $\Gamma \in C^{2,\lambda}$, it then follows from [3, Lemmas 2, 3] and [4, Lemmas 1, 3] that $v_j(s,\sigma) \in C^{0,p_0}(\Gamma\times\Gamma)$, $j = 1,\dots,4$, where

$$p_0 = \begin{cases}\lambda & \text{if } 0<\lambda<1,\\ 1-\varepsilon_0 & \text{for any } \varepsilon_0 \in (0,1] \text{ if } \lambda = 1.\end{cases} \tag{13}$$

We put $c = \sqrt{\beta^2+1}$, $c_\pm = c \pm \beta = \sqrt{\beta^2+1} \pm \beta$. Substituting the unknown functions by the formulas

$$\rho_1(s) = c_-\nu(s) - \mu(s) \in C^r_\varkappa(\Gamma), \quad \rho_2(s) = c_-\mu(s) + \nu(s) \in C^r_\varkappa(\Gamma), \tag{14}$$

$$\nu(s) = (\rho_1(s) + c_+\rho_2(s))/(2c), \quad \mu(s) = (\rho_2(s) - c_+\rho_1(s))/(2c), \tag{15}$$

we get equations (12) in the new variables ($s \in \Gamma$)

$$\begin{gathered}\rho_1(s) - \frac{c_+}{\pi}\int\limits_\Gamma \rho_1(\sigma)\frac{d\sigma}{\sigma - s} + \int\limits_\Gamma \rho_1(\sigma)Y_{11}(s,\sigma)\,d\sigma + \int\limits_\Gamma \rho_2(\sigma)Y_{12}(s,\sigma)\,d\sigma\\ = 2(cf'^{+}(s) + f^{-}(s)) \equiv f_1(s) \in C^{0,\lambda}(\Gamma),\end{gathered}$$

$$\rho_2(s)+\frac{c_-}{\pi}\int\limits_\Gamma \rho_2(\sigma)\frac{d\sigma}{\sigma-s}+\int\limits_\Gamma \rho_1(\sigma)Y_{21}(s,\sigma)\,d\sigma+\int\limits_\Gamma \rho_2(\sigma)Y_{22}(s,\sigma)\,d\sigma \\ =2c_-(cf'^+(s)-f^-(s))\equiv f_2(s)\in C^{0,\lambda}(\Gamma). \tag{16}$$

Here $Y_{11}(s,\sigma)=\{cv_1(s,\sigma)-c_+(cv_2(s,\sigma)+v_3(s,\sigma))-v_4(s,\sigma)\}/(2c)$, $Y_{12}(s,\sigma)=\{c_+cv_1(s,\sigma)+cv_2(s,\sigma)+v_3(s,\sigma)-c_+v_4(s,\sigma)\}/(2c)$, $Y_{21}(s,\sigma)=\{c_-cv_1(s,\sigma)-cv_2(s,\sigma)+v_3(s,\sigma)+c_-v_4(s,\sigma)\}/(2c)$, $Y_{22}(s,\sigma)=\{cv_1(s,\sigma)+c_-(cv_2(s,\sigma)-v_3(s,\sigma))+v_4(s,\sigma)\}/(2c)$. It follows from the smoothness of the functions $v_1(s,\sigma),\dots,v_4(s,\sigma)$ that $Y_{jl}(s,\sigma)\in C^{0,p_0}(\Gamma\times\Gamma)$, $j,l=1,2$; p_0 is borrowed from (13).

In terms of $\rho_1(s)$, $\rho_2(s)$, conditions (9), (11) will assume the form

$$V\left[\rho_2(s)-c_+\rho_1(s)\right](x(a_n))+T\left[\rho_1(s)+c_+\rho_2(s)\right](x(a_n))=2cf^+(a_n), \\ \int\limits_{a_n}^{b_n}(\rho_1(s)+c_+\rho_2(s))\,d\sigma=0,\quad n=1,\dots,N. \tag{17}$$

System (16), (17) is a particular case of the systems studied in [5]. By Theorem 1 from [5], all the solutions $\rho_1(s)$, $\rho_2(s)$ of system (16), (17) that belong to $C^r_\varkappa(\Gamma)$, $r\in(0,1]$, $\varkappa\in[0,1)$, are representable in the form $\rho_j(s)=\rho_{j*}(s)/Q_j(s)$, $j=1,2$, where $\rho_{1*}(s),\rho_{2*}(s)\in C^{0,\omega}(\Gamma)$, $\omega=\min\{\lambda,\gamma,\frac12-\gamma\}$, $Q_1(s)=\prod_{n=1}^N|s-a_n|^{1/2-\gamma}|s-b_n|^{1/2+\gamma}\operatorname{sign}(s-a_n)\in C^{0,1/2-\gamma}(\Gamma)$, $Q_2(s)=\prod_{n=1}^N|s-a_n|^{1-\gamma}|s-b_n|^{\gamma}\operatorname{sign}(s-a_n)\in C^{0,\gamma}(\Gamma)$; the number γ is defined by the equality $\gamma=\frac{1}{2\pi}\arctan\beta\in(0,\frac12)$. Hence it follows that $\rho_1(s),\rho_2(s)\in C^{\overline\omega}_q(\Gamma)$, where

$$q=\max\left\{\frac12+\gamma,\,1-\gamma\right\},\quad \overline\omega=\begin{cases}\min\left\{\omega,\left|\frac12-2\gamma\right|\right\} & \text{if } \gamma\neq\frac14,\\ \omega & \text{if } \gamma=\frac14.\end{cases} \tag{18}$$

Let us prove that homogeneous system (16), (17) has only a trivial solution among the functions $\rho_1(s),\rho_2(s)\in C^{\overline\omega}_q(\Gamma)$. Let $\rho_1^0(s)$, $\rho_2^0(s)$ be the solution of homogeneous system (16), (17) in the space $C^{\overline\omega}_q(\Gamma)$, where $\overline\omega$ and q are borrowed from (18). In that case, the functions $\nu^0(s)=(\rho_1^0(s)+c_+\rho_2^0(s))/(2c)$, $\mu^0(s)=(\rho_2^0(s)-c_+\rho_1^0(s))/(2c)$ constructed by formulas (15) give the solution of homogeneous system (9)–(11). Obviously, homogeneous Problem $\mathcal U$ (with $f^+(s)\equiv f^-(s)\equiv 0$) is associated with homogeneous system (9)–(11). According to Theorem 2, the function $u^0(x)=V[\mu^0](x)+T[\nu^0](x)$ is the solution of homogeneous Problem $\mathcal U$. Using Theorem 1, we have $u^0(x)\equiv 0$. From the limiting formulas for the tangential and normal derivatives of potentials [3] we obtain

$$\mu^0(s)=\left.\frac{\partial u^0}{\partial\mathbf n_x}\right|_{x(s)\in\Gamma^+}-\left.\frac{\partial u^0}{\partial\mathbf n_x}\right|_{x(s)\in\Gamma^-}\equiv 0,\quad \nu^0(s)=\left.\frac{\partial u^0}{\partial\tau_x}\right|_{x(s)\in\Gamma^+}-\left.\frac{\partial u^0}{\partial\tau_x}\right|_{x(s)\in\Gamma^-}\equiv 0,$$

hence $\rho_1^0(s)\equiv\rho_2^0(s)\equiv 0$. Homogeneous system (16), (17) thereby has only a trivial solution among the functions $\rho_1^0(s),\rho_2^0(s)\in C^{\overline\omega}_q(\Gamma)$, where $\overline\omega$ and q are borrowed from (18). The following theorem issues from Theorem 1 (item 1) and Theorem 2 from [5].

Theorem 3. *Let* $\Gamma\in C^{2,\lambda}$, $f^+(s)\in C^{1,\lambda}(\Gamma)$, $f^-(s)\in C^{0,\lambda}(\Gamma)$, $\lambda\in(0,1]$. *In that case, inhomogeneous system* (16), (17) *has the only solution among the functions* $\rho_1(s),\rho_2(s)\in C^r_\varkappa(\Gamma)$ ($r\in(0,1]$, $\varkappa\in[0,1)$). *This solution is representable as* $\rho_j(s)=\rho_{j*}(s)/Q_j(s)$, $j=1,2$, *where* $\rho_{1*}(s),\rho_{2*}(s)\in C^{0,\omega}(\Gamma)$, $\omega=\min\{\lambda,\gamma,\frac12-\gamma\}$; *consequently,* $\rho_1(s),\rho_2(s)\in C^{\overline\omega}_q(\Gamma)$ ($\overline\omega$ *and* q *are borrowed from* (18)).

The solution $\mu(s)$, $\nu(s)\in C^{\overline\omega}_q(\Gamma)$ of system (9)–(11) is defined by formulas (15) where $\rho_1(s)$, $\rho_2(s)$ is the solution of system (16), (17), guaranteed by Theorem 3. One more theorem follows from Theorems 1 and 2.

Theorem 4. *Let* $\Gamma\in C^{2,\lambda}$, $f^+(s)\in C^{1,\lambda}(\Gamma)$, $f^-(s)\in C^{0,\lambda}(\Gamma)$, $\lambda\in(0,1]$. *In that case, the solution of Problem* $\mathcal U$ *exists, is unique, and is given by formula* (8) *where the densities* $\mu(s)$, $\nu(s)$ *are taken from* (15), *where the functions* $\rho_1(s),\rho_2(s)\in C^{\overline\omega}_q(\Gamma)$ ($\overline\omega$ *and* q *are specified by expressions* (18)) *are the solution of system* (16), (17), *guaranteed by Theorem* 3.

Theorem 4 establishes the existence and uniqueness of the classical solution of Problem $\mathcal U$. As follows from the definition of the class $\mathcal G$, the gradient of the solution of Problem $\mathcal U$ can have singularities at the ends of the contour Γ. According to Theorem 5 (item 3) from [3], inequality (1) holds true with $\delta = -q$. We write the explicit asymptotic formulas describing the singularities ∇u at the ends of the contour Γ. Let $x(d)$ be one of the ends of the contour Γ, $d = a_n$ or $d = b_n$, $n = 1, \dots, N$, i.e., $x(d) \in X$. We introduce in the neighborhood of $x(d)$ the polar coordinate system $x_1 = |x - x(d)| \cos\varphi$, $x_2 = |x - x(d)| \sin\varphi$. Recall that $\alpha(s)$ is the angle between the direction of the axis Ox_1 and the vector of the tangent line τ_x at the point $x(s) \in \Gamma$. Considering that $\varphi \in (\alpha(d), \alpha(d) + 2\pi)$ if $d = a_n$ and $\varphi \in (\alpha(d) - \pi, \alpha(d) + \pi)$ if $d = b_n$, $n = 1, \dots, N$, we assume from continuity that $\alpha(a_n) = \alpha(a_n + 0)$, $\alpha(b_n) = \alpha(b_n - 0)$ and introduce the functions $\hat\rho_1(a_n, s) = \rho_1(s)|s - a_n|^{1/2-\gamma}$, $\hat\rho_2(a_n, s) = \rho_2(s)|s - a_n|^{1-\gamma}$ that are Hoelder functions on Γ in the neighborhood of a_n and the functions $\hat\rho_1(b_n, s) = \rho_1(s)|s - b_n|^{1/2+\gamma}$, $\hat\rho_2(b_n, s) = \rho_2(s)|s - b_n|^{\gamma}$ that are Hoelder functions on Γ in the neighborhood of b_n. Repeating the reasoning from [6], we get the following theorem.

Theorem 5. *Let $x \to x(d) \in X$, where $d = a_n$ or $d = b_n$, $n = 1, \dots, N$. In that case, in the neighborhood of the point $x(d)$ the following formulas hold true for the derivatives of the solution of Problem $\mathcal{U}$:*

(1) *at $d = a_n$,*

$$\left.\frac{\partial u}{\partial x_j}\right|_{x \to x(a_n)} = \frac{\hat\rho_2(a_n, a_n)}{2|x - x(a_n)|^{1-\gamma}} c_+ \cos\left[j\frac{\pi}{2} - \theta(a_n, 1-\gamma)\right] - \frac{\hat\rho_1(a_n, a_n)}{2|x - x(a_n)|^{1/2-\gamma}} \cos\left[j\frac{\pi}{2} - \theta\left(a_n, \frac{1}{2} - \gamma\right)\right] + O(1),$$

(2) *at $d = b_n$,*

$$\left.\frac{\partial u}{\partial x_j}\right|_{x \to x(b_n)} = \frac{\hat\rho_2(b_n, b_n)}{2|x - x(b_n)|^{\gamma}} c_+ \cos\left[j\frac{\pi}{2} - \theta(b_n, \gamma)\right] + \frac{\hat\rho_1(b_n, b_n)}{2|x - x(b_n)|^{1/2+\gamma}} \sin\left[j\frac{\pi}{2} - \theta\left(b_n, \frac{1}{2} + \gamma\right)\right] + O(1),$$

where $j = 1, 2$; $\theta(a_n, \eta) = \eta\varphi + (1 - \eta)\alpha(a_n)$; $\theta(b_n, \eta) = \eta\varphi + (1 - \eta)\alpha(b_n) - \pi\gamma$; the symbol $O(1)$ denotes functions continuous in the neighborhood of the point $x(d)$ that is cut along the contour Γ. Moreover, the functions designated as $O(1)$ are continuous at the very point $x(d)$.

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17 May 2004 Department of Mathematics