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PARQUET APPROXIMATION FOR MATRIX MODELS

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Application of the parquet-planar approximation to description of a zero-dimensional two-matrix model is considered.

1. In the present work a parquet-planar approximation for the zero-dimensional two-matrix model is constructed. We first recall the basics of the models and methods used here.

The parquet approximation is constructed basing upon two different approaches employed in the modern quantum field theory.

The parquet approximation was proposed in a series of publications by Landau, Khalatnikov and Abrikosov as one of the first nonperturbative methods to describe quantum electrodynamics [1]. This approximation is defined using the system of integro-differential equations for propagators and vertex functions. The complete theory is constructed in terms of solutions of this system. The main feature of this approach is that the parquet system of equations is valid for both small and large values of the coupling constant. The parquet approximation proved to be a convenient tool for studying various physical models [2–5].

The planar approximation is based on the following observation. It turns out that in certain matrix field theories, for instance in QCD with the $SU(N)$ gauge group, the perturbation theory can be constructed with expansion in powers of $1/N$ rather than in powers of the coupling constant. Then the perturbation theory expansion series has a topological nature, the main contribution being provided by the so-called planar diagrams, i. e., the diagrams that can be mapped to the plane without self-crossing [6].

The two approaches have both merits and demerits. For instance, the parquet approximation, as opposed to the planar one, violates gauge invariance of the theory. At the same time, the parquet approximation is applicable in spaces of arbitrary dimensions, while the planar approximation can hardly be used in spaces with dimensions different from 0 or 1.

This is why the parquet-planar approximation was aimed at attempting to combine the advantages and possibly to eliminate shortcomings of two approaches.

2. The parquet-planar approximation was proposed in [7]. The so-called zero-dimensional one-matrix model considered had the action

$$S(M) = \frac{1}{2}\,\mathrm{Tr}\,M^2 + \frac{g}{4N}\,\mathrm{Tr}\,M^4,$$

where M is a Hermitian $N \times N$ matrix.

This model is one of the simplest examples of polynomial models that play an important role in modern physics and mathematics.

Defining the Green functions as

$$\Pi_n = <\mathrm{Tr}\,M^n> \equiv \lim_{N\to\infty} \mathcal{N}\frac{1}{N^{1+n/2}} \int DM\ \mathrm{Tr}\,M^n\, e^{-S(M)},$$

we can show that they satisfy the planar Schwinger–Dyson equations of the form

$$\Pi_n + g\Pi_{n+2} = \sum_{i=0}^{n-2} \Pi_i \Pi_{n-i-2}, \quad n \geq 2. \tag{1}$$

These equations form an infinite chain, and it can be demonstrated that the complete set of Green's functions can be obtained using a generating functional depending on a parameter, and Π_2 can be chosen as this parameter. In order to define it, a supplementary condition should be set. Thus, a possibility arises to control the approximation in a certain way.

The parquet-planar approximation is defined by means of the following system of equations:

$$\begin{cases} \Pi_2 = 1 - 2g\Pi_2^2 - g\Pi_2^4\Gamma_4 = 1 - g\Pi_4, \\ \Gamma_4 = -g + H + V, \\ H = -g\Pi_2^2\Gamma_4 + V\Pi_2^2\Gamma_4, \\ V = -g\Pi_2^2\Gamma_4 + H\Pi_2^2\Gamma_4, \end{cases} \tag{2}$$

where Γ_4 is a four-point vertex function, H (V) describes the part of it that is two-particle reducible in the t-channel (s-channel) and two-particle irreducible in the s-channel (t-channel). The first equation of the system is the Schwinger–Dyson equation. The remaining equations are called the parquet equations. Thus, the parquet-planar approximation is the way to close the infinite chain (1) and to write the auxiliary equation necessary to completely define the theory. Planarity of the solution of system (2) follows from the fact that we omit the contribution of the u-channel to the vertex function*.

Since the zero-dimensional case is considered, system (2) is pure algebraic, and its solution reproduces with high accuracy the "exact" planar results of the classical paper [8].

Table 1

	Planar approximation	Parquet-planar approximation
$\Pi_2(g)$ for small g	$1 - 8g + o(g)$	$1 - 8g + o(g)$
$\Pi_2(g)$ for large g	$0.7698g^{-1/2}$	$0.7695g^{-1/2}$
g_{crit}	-0.083	-0.084

3. Another important class of Hermitian matrix models are the so-called two-matrix models [9, 10]. The simplest of them is determined by the action

$$S(M_1, M_2) = \operatorname{Tr} M_1^2 + \frac{g}{N} \operatorname{Tr} M_1^4 + \operatorname{Tr} M_2^2 + \frac{g}{N} \operatorname{Tr} M_2^4 - 2c \operatorname{Tr} M_1 M_2.$$

This model has proved to be equivalent to the Ising model on random lattice [11]. The more nontrivial consequence of the model is that it can be extended to the case of several matrices with chain-like interaction $\sum_{i=1}^{p-1} c_i \operatorname{Tr} M_i M_{i+1}$ [12], then in the limit $p \to \infty$, the initial model goes over to matrix quantum mechanics [13].

We shall briefly recall the basic equations and results of the two-matrix model.

The partition function is defined in the following way:

$$Z = \int DM_1\, DM_2\, e^{-S(M_1,M_2)} = \text{const} \int \prod_{i=1}^{N} dx_i dy_i\, \Delta(x_i)\Delta(y_i) e^{-S(x_i,y_i)}, \quad \Delta(x_i) = \prod_{i<j=1}^{N} (x_i - x_j)^2.$$

Here M_1 and M_2 are reduced to the diagonal form, $X = \operatorname{diag}(x_i)$ and $Y = \operatorname{diag}(y_i)$.

* In terms of the Feynman diagrams, the parquet-planar approximation means the following. System of equations (2) accounts for the parquet-planar diagrams, i. e., the parquet diagrams that can be mapped on the plane without self-crossing. As for the parquet diagrams, they are constructed with the help of the skeleton approximation using the vertex functions that include the diagrams two-particle irreducible in the s- and t-channels (the contribution of the u-channel is excluded in virtue of its nonplanarity).

The matrix models of this form are studied with the help of the orthogonal polynomial technique [10]. In the case of a two-matrix model, a set of orthogonal and biorthogonal polynomials $P_i(x)$ is introduced,

$$\int_{-\infty}^{\infty} P_i(x)P_j(y)e^{-S(x,y)}\,dx\,dy = h_i\delta_{ij}, \quad S(x,y) = x^2 + \frac{gx^4}{N} + y^2 + \frac{gy^4}{N} - 2cxy.$$

Polynomials $P_i(x)$ satisfy the recurrent relation

$$xP_i(x) = P_{i+1}(x) + R_iP_{i-1}(x) + T_iP_{i-3}(x),$$

hence, with the use of the orthogonality condition, we can obtain a system of equations for the coefficients R_i, T_i and $f_i \equiv h_i/h_{i-1}$

$$\begin{cases} f_i = \dfrac{cR_i}{1 + \frac{2g}{N}(R_{i+1} + R_i + R_{i-1})}, \\ cf_i = -i/2 + R_i\Big(1 + \dfrac{2g}{N}(R_{i+1} + R_i + R_{i-1})\Big) + \dfrac{2g}{N}(T_{i+2} + T_{i+1} + T_i), \\ cT_i = \dfrac{2g}{N}f_if_{i-1}f_{i-2}. \end{cases} \tag{3}$$

In the limit of large N, coefficients R_i, T_i and f_i determine the continuous functions

$$\frac{i}{N} \to x \in [0,1], \quad \frac{f_i}{N} \to f(x), \quad \frac{R_i}{N} \to R(x), \quad \frac{T_i}{N^2} \to T(x),$$

and system (3) takes the form

$$\begin{cases} f(x) = cR(x)(1 + 6gR(x))^{-1}, \\ cf(x) = -x/2 + R(x)(1 + 6gR(x)) + 6gT(x), \\ cT(x) = 2gf^3(x). \end{cases}$$

Then the partition function is defined by the simple expression

$$\ln Z = \ln\left(\text{const}\prod_{i=0}^{N-1} h_i\right) \to \text{const} + \int_0^1 (1-x)\ln f(x)\,dx. \tag{4}$$

We calculate the two-point correlation function $D_1 = <M_1^2> \equiv <M_2^2> = D_2$,

$$D_1 = \lim_{N\to\infty} \mathcal{N}\int dM_1\,dM_2\,\mathrm{Tr}\,M_1^2 e^{-S(M_1,M_2)} = f(0) + 2\int_0^1 (1-x)f'(x)\,dx. \tag{5}$$

Then its behavior at small g is defined by the expression

$$D_1 = \frac{1}{2(1-c^2)} - \frac{g(1+c^2)}{(1-c^2)^3} + \frac{g^2(3c^2+9)(2c^2+1)}{2(1-c^2)^5} + O(g^3). \tag{6}$$

4. We define the parquet-planar approximation for the zero-dimensional matrix model as a solution of the following system of equations.

The first four equations are the Schwinger–Dyson equations for propagators $D_1 = <\mathrm{Tr}\,M_1^2>$, $D_2 = <\mathrm{Tr}\,M_2^2>$, effective vertex $\Pi_2 = <\mathrm{Tr}\,M_1M_2>$ which describes interconversion of matrix fields and four-point vertices Γ_1 and Γ_2 that correspond to each of the matrix fields (mixed vertices will not be considered),

$$\begin{cases} D_1 = 1 - 8gD_1^2 - 4gD_1^4\Gamma_1 + c\Pi_2, \\ D_2 = 1 - 8gD_2^2 - 4gD_2^4\Gamma_2 + c\Pi_2, \\ \Pi_2 = cD_1 - 8gD_2\Pi_2, \\ \Pi_2 = cD_2 - 8gD_1\Pi_2. \end{cases} \tag{7}$$

Vertices Γ_1 and Γ_2 are defined by the system

$$\begin{cases} \Gamma_1 = -4g + H_1 + V_1, \\ H_1 = -4gD_1^2\Gamma_1 + V_1D_1^2\Gamma_1, \\ V_1 = -4gD_1^2\Gamma_1 + H_1D_1^2\Gamma_1, \\ \Gamma_2 = -4g + H_2 + V_2, \\ H_2 = -4gD_2^2\Gamma_2 + V_2D_2^2\Gamma_2, \\ V_2 = -4gD_2^2\Gamma_2 + H_2D_2^2\Gamma_2. \end{cases} \tag{8}$$

Here H_1 and V_1 (H_2 and V_2) describe the contributions to the four-point vertex functions that are two-particle reducible with respect to matrix field M_1 (M_2) in t-and s-channels.

System of equations (7), (8) can be solved in the limit of small g. The asymptotic expression for the two-point function has the form

$$D_{1,2} = \frac{1}{2(1-c^2)} - \frac{g(1+c^2)}{(1-c^2)^3} + \frac{g^2(2c^2+1)(2c^2+9)}{2(1-c^2)^5} + O(g^3). \tag{9}$$

It is seen from (6) and (9) that in the region of small g, the parquet-planar approximation reproduces the results of the planar approximation.

From system (7), (8), we can obtain an equation only for D_1 or for D_2, g and c. This equation is the eighth order in D_1 and determines D_1 as a function of coupling constants. The graph of this function has two sheets in the physically interesting region $0 \leq c \leq 1$, and the transition from one sheet to another is associated with the phase transition in the corresponding statistical system.

We note that if c vanishes, which corresponds to the case of two independent systems, the solution goes over to the already known solution of the one-matrix model proposed in [7].

5. In conclusion, we discuss possible development of the ideas proposed above.

First, as was already pointed out, application of both the parquet and planar approaches to the study of matrix quantum mechanics is of interest. The method of investigation differs from the standard one. It is proposed to start with a finite approximation of quantum mechanics by means of the p-matrix model and then to study the limit of large p, and this in its turn should give a description of quantum mechanics.

Second, as it is known, the parquet approach is usually applied only in the limit of large N, i. e., it describes the planar or spherical limit. It is proposed to investigate if the approach is applicable for the description of the case of a more general topology.

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