Moscow University
Physics Bulletin
Vol. 60, No. 5, pp. 10–14, 2005

Vestnik Moskovskogo
Universiteta. Fizika
UDC 621.372.2

A HIGH-ORDER FINITE-ELEMENT METHOD FOR CYLINDRICAL WAVEGUIDES

A. L. Delitsyn and I. V. Stepanov

E-mail: delitsin@math356.phys.msu.ru; veryhot@hotmail.ru

A method for calculating the modes of gradient insulator waveguides is developed. The results of testing this method for hollow and double-layer waveguides suggest that the method is highly accurate. The proposed method makes it possible to solve the problem of approximating the field strength at the origin of a cylindrical system of coordinates.

Two problems emerge when the finite-element method is used to calculate an insulator waveguide in a cylindrical system of coordinates: the problem of approximating the field at the origin of the system of coordinates, and the problem of approximating a zero eigenvalue of infinite multiplicity [1]. One way to solve the first problem is to introduce a fictitious additional boundary near the origin of the cylindrical system of coordinates, a boundary that corresponds to an ideally conducting rod of a small radius. As the rod's radius is sent to zero, the eigenvalues of the problem in question tend to the eigenvalues of the initial problem. Samarskii [2] proved this for the case of the natural frequencies of a membrane, and we use his approach as a base for our method.

Let us take a cylindrical waveguide with a circular cross-sectional area Ω of unit radius. At a certain point on the waveguide's axis we "place" the origin of a cylindrical system of coordinates whose Oz axis coincides with the cylinder's axis. Suppose that the waveguide is filled with substance whose characteristics are

$$\varepsilon(r, \varphi, z) = \varepsilon(r), \quad \mu(r, \varphi, z) = 1$$

with ε piecewise continuous and the walls ideal conductors.

We look for the solution of the Maxwell equations in the form of normal modes $E, H \sim e^{-i\omega t + i\gamma z + im\varphi}$. After canceling out the exponential factor, we arrive at the problem on the interval $[0, 1]$, with the parameter γ acting as the eigenvalue. Following [3], we select from the system of eight Maxwell equations six main equations for the six unknowns in the following way:

$$\begin{cases} \dfrac{1}{r}\dfrac{d}{d\varphi}H_z - i\gamma H_\varphi + ik\varepsilon E_k = 0, \\ i\gamma H_r - \dfrac{d}{dr}H_z + ik\varepsilon E_\varphi = 0, \\ \dfrac{1}{r}\dfrac{d}{dr}(r\varepsilon E_r) + \dfrac{1}{r}\dfrac{d}{d\varphi}(\varepsilon E_\varphi) + i\gamma\varepsilon E_z = 0, \end{cases} \tag{1}$$

$$\begin{cases} \dfrac{1}{r}\dfrac{d}{d\varphi}E_z - i\gamma E_\varphi - ikH_r = 0, \\ i\gamma E_r - \dfrac{d}{dr}E_z - ikH_\varphi = 0, \\ \dfrac{1}{r}\dfrac{d}{dr}(rH_r) + \dfrac{1}{r}\dfrac{d}{d\varphi}H_\varphi + i\gamma H_z = 0. \end{cases} \tag{2}$$

We introduce the notation $X = (H_r, H_\varphi, E_z)^T = (H_\perp, E_z)^T$ and $Y = (\varepsilon E_r, \varepsilon E_\varphi, H_z)^T = (\varepsilon E_\perp, H_z)^T$. Plugging Y from (2) into (1), we arrive at the eigenvalue equation

$$\begin{pmatrix} r\dfrac{d}{dr}\dfrac{1}{r}\dfrac{d}{dr}r + k^2\varepsilon r & imr\dfrac{d}{dr}\dfrac{1}{r} & -k\varepsilon m \\ im\dfrac{1}{r}\dfrac{d}{dr}r & -\dfrac{m^2}{r} + k^2\varepsilon r & -ik\varepsilon r\dfrac{d}{dr} \\ km\varepsilon & ik\dfrac{d}{dr}r\varepsilon & \dfrac{d}{dr}r\varepsilon\dfrac{d}{dr} - \dfrac{\varepsilon}{r}m^2 \end{pmatrix} X = \gamma^2 r \begin{pmatrix} 1 & & \\ & 1 & \\ & & \varepsilon \end{pmatrix} X, \tag{3}$$

where X belongs to the set of vectors from $C^\infty[0,1]$. We augment equation (3) by the boundary conditions

$$H_r(1) = 0 \quad \text{and} \quad E_z(1) = 0, \quad |X(0)| < \infty \tag{4}$$

and the Maxwell equation not employed earlier,

$$-ik\varepsilon E_z = \frac{1}{r}\frac{d}{dr}(rH_\varphi) - \frac{1}{r}inH_r. \tag{5}$$

We introduce the operators

$$\operatorname{grad}\varphi(r) = \left(\frac{d}{dr}\varphi, \frac{im}{r}\varphi\right)^T, \quad \operatorname{curl}\varphi(r) = \left(\frac{im}{r}\varphi, -\frac{d}{dr}\varphi\right)^T, \quad \operatorname{div} H_\perp = \frac{1}{r}\frac{d}{dr}rH_r + \frac{im}{r}H_\varphi.$$

The distribution of the eigenvalues in the complex plane and their asymptotic behavior were obtained in [3].

Let us write the variational functional for our problem. To this end we multiply (3) on the left by an arbitrary vector $\overline{Y}^T$ and integrate the product with respect to r from 0 to 1. The result is

$$\begin{aligned} &\int_0^1 r\,dr \operatorname{div} H_\perp \overline{\operatorname{div} H_\perp} + \int_0^1 \varepsilon r\,dr\left(\operatorname{grad} E_z, \overline{\operatorname{grad} E_z}\right) \\ &- \int_0^1 dr\Big\{k^2\varepsilon r\left(\overline{H_r}H_r + \overline{H_\varphi}H_\varphi\right) - k\varepsilon m\overline{H_r}E_z + km\varepsilon\overline{E_z}H_r - ik\varepsilon r\overline{H_\varphi}\frac{d}{dr}E_z - ik\varepsilon rH_\varphi\frac{d}{dr}\overline{E_z}\Big\} \\ &= -\gamma^2\int_0^1 r\,dr\left(H_r\overline{H_r} + H_\varphi\overline{H_\varphi} + \varepsilon E_z\overline{E_z}\right). \end{aligned} \tag{6}$$

Since the problem (3), (4) has a singularity at zero, it is difficult to solve it numerically by the finite-element method. This fact prompted us to replace it with a new problem, in which a conducting cylinder of a small radius a is placed near the origin. Mathematically, this amounts to studying the problem

$$\begin{aligned} &\int_a^1 r\,dr \operatorname{div} H_\perp \overline{\operatorname{div} H_\perp} + \int_a^1 \varepsilon r\,dr\left(\operatorname{grad} E_z, \overline{\operatorname{grad} E_z}\right) \\ &- \int_a^1 dr\Big\{k^2\varepsilon r\left(\overline{H_r}H_r + \overline{H_\varphi}H_\varphi\right) - k\varepsilon m\overline{H_r}E_z + km\varepsilon\overline{E_z}H_r - ik\varepsilon r\overline{H_\varphi}\frac{d}{dr}E_z - ik\varepsilon rH_\varphi\frac{d}{dr}\overline{E_z}\Big\} \\ &= -\gamma^2\int_a^1 r\,dr\left(H_r\overline{H_r} + H_\varphi\overline{H_\varphi} + \varepsilon E_z\overline{E_z}\right) \end{aligned} \tag{7}$$

on the interval $[a, 1]$, introducing the boundary conditions

$$H_r(1) = 0 \quad \text{and} \quad E_z(1) = 0, \quad |X(a)| < \infty \tag{8}$$

with X belonging to the set of vectors from $C^\infty[q,1]$ that satisfy the boundary conditions, and employing the Maxwell equation not used earlier,

$$-ik\varepsilon E_z = \frac{1}{r}\frac{d}{dr}(rH_\varphi) - \frac{1}{r}imH_r. \tag{9}$$

To simplify the computations, we introduce the notation $rH_r \equiv H_r$ and $iH_\varphi \equiv H_\kappa$.

We find the solution by a method that became known as the mixed finite-element method [4]. On the interval $r \in [a,1]$ we introduce a mesh $\{x_j\}$: $j = 0,\ldots,N$, $x_0 = a$, $x_N = 1$ with N being the number of finite elements.

We expand the fields in base functions with finite support functions,

$$X_i = \sum_{n=0}^{N_i} c_{in}\varphi_{in}, \quad i = 1,2,3,$$

where N_i is the number of base functions, which depends on the number of finite elements and their order, and c_{in} are the unknown coefficients. Here for H_φ we take first-order discontinuous elements

$$\varphi_{2n} = \begin{cases} 0, & x \le x_j, \\ \dfrac{x_{j+1}-x}{x_{j+1}-x_j}, & x_j \le x \le x_{j+1}, \\ 0, & x \ge x_{j+1}, \end{cases} \qquad \varphi_{2(n+1)} = \begin{cases} 0, & x \le x_j, \\ \dfrac{x-x_j}{x_{j+1}-x_j}, & x_j \le x \le x_{j+1}, \\ 0, & x \ge x_{j+1}. \end{cases}$$

For H_r and E_z we take second-order continuous elements

$$\varphi_{3,4n} = \begin{cases} 0, & x \le x_j, \\ -\dfrac{4(x-x_j)(x-x_{j+1})}{(x_{j+1}-x_j)^2}, & x_j \le x \le x_{j+1}, \\ 0, & x \ge x_{j+1}, \end{cases}$$

$$\varphi_{3,4(n+1)} = \begin{cases} 0, & x \le x_j, \\ \dfrac{2(x-x_j)(x-0.5(x_j+x_{j+1}))}{(x_{j+1}-x_j)^2}, & x_j \le x \le x_{j+1}, \\ \dfrac{2(x-x_{j+1})(x-0.5(x_j+x_{j+1}))}{(x_{j+1}-x_j)^2}, & x_{j+1} \le x \le x_{j+2}, \\ 0, & x \ge x_{j+2}. \end{cases}$$

Here at the points x_0 and x_N the base functions must satisfy the boundary conditions of the initial problem.

Plugging the above expansions into the variational functional and integrating, we arrive at the following eigenvalue problem: $A(k,m) = -\gamma^2 B$, where $A(k,m) = a(k,m)_l^l$ and $B = b_l^l$, $l = 0,\ldots,N_1+N_2+N_3$ are sparse square matrices.

A program for calculating cylindrical waveguides with arbitrary piecewise continuous filling with an insulating material has been compiled. The testing was carried out for the case of hollow and double-layer waveguides, which can be solved exactly.

In the case of a hollow waveguide, the system of waves is a linear combination of fields of the electric and magnetic types. The solution can be expressed in terms of the Hertz vector $\Pi = CJ_m(\lambda r)\cos m\varphi$ or $\Pi = CJ_m(\lambda r)\sin m\varphi$. Then the eigenvalue equation for the electric waves has the form $\Pi^e = 0$ at $r = 1$, i. e, $J_m(\lambda) = 0$, with λ_{mn} being the nth positive root. For magnetic waves the equation is $d\Pi^m/dr = 0$ at $r = 1$, i. e., $J'_m(\lambda) = 0$, with λ_{mn} being the nth positive root.

Table 1 lists the calculated eigenvalues and the exact eigenvalues (in parentheses) for 40 finite elements and the inner rod radius $a = 10^{-6}$.

Table 2 illustrates the dependence of the eigenvalues of the zeroth mode on the inner radius a for 20 finite elements.

Table 1

Mode number	Number of γ	γ	
		TM wave	TE wave
0	1	2.4048256 (2.4048261)	3.8317064 (3.8317061)
	2	5.5200791 (5.5200780)	7.0155942 (7.0155858)
	3	8.6537380 (8.6537277)	10.173511 (10.1734682)
1	1	3.8317062 (3.8317061)	1.8409158 (1.8411838)
	2	7.0155895 (7.0155858)	5.3298359 (5.3314429)
	3	10.173489 (10.1734682)	8.5322647 (8.5363167)
2	1	5.1356226 (5.13562231)	3.0542370 (3.05423697)
	2	8.4172498 (8.41724411)	6.7061354 (6.70613322)
	3	11.619875 (11.6198408)	9.9694866 (9.96946724)

Table 2

Exact eigenvalues	Inner conducting rod radius a					
	$a = 10^{-2}$	$a = 10^{-3}$	$a = 10^{-4}$	$a = 10^{-5}$	$a = 10^{-6}$	$a = 10^{-7}$
2.4048261	2.4052714	2.4048302	2.4048258	2.4048257	2.4048257	2.4048257
5.5200780	5.5224651	5.5201168	5.5200932	5.5200930	5.5200930	5.5200930
3.8317061	3.8328913	3.8317252	3.8317135	3.8317134	3.8317134	3.8317134
7.0155858	7.0195663	7.0157455	7.0157080	7.0157080	7.0157079	7.0157081

These results suggest that the use high-order finite elements makes it possible to approximate the field strength at the origin and leads to high accuracy even for a coarse mesh.

Now let us turn to the case of a double-layer waveguide with an axisymmetric filling of radius b and a dielectric constant ε. The modes of round layered waveguides exhibit a number of features that waves in homogeneously filled waveguides and axisymmetric waves do not have. In particular, such double-layer waveguides feature anomalous dispersion ($d\gamma/dk < 0$). For certain values of the waveguide parameters, these waves (even in the absence of energy dissipation) are transformed into waves with complex-valued propagation constants, or complex waves.

We seek the solution by the method of mixed finite elements described above. The calculation used 20 finite elements and an inner rod with a radius $a = 10^{-6}$.

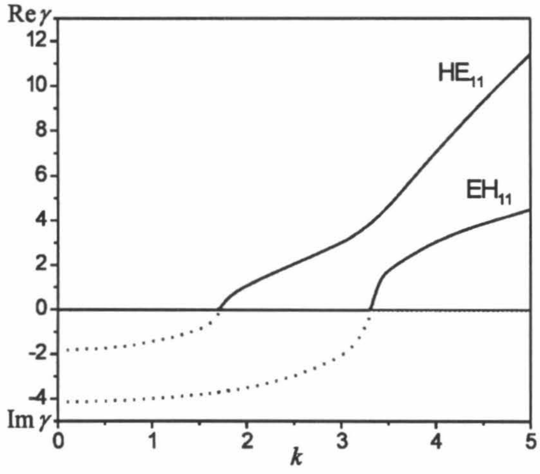

Fig. 1

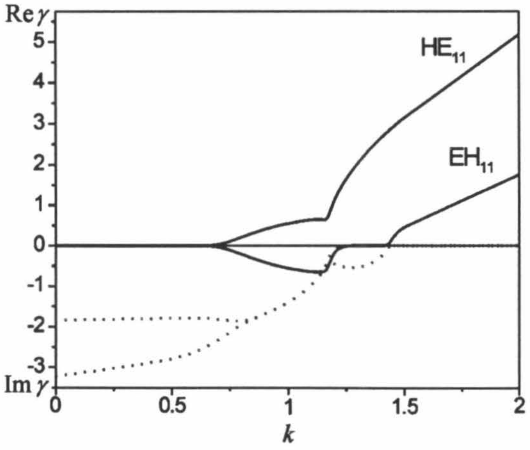

Fig. 2

Figures 1 and 2 demonstrate the dispersion characteristics of the lower types of hybrid waves HE_{11} and EH_{11} in a waveguide with $\varepsilon = 10$ and $b = 0.2$ (Fig. 1) and $b = 0.6$ (Fig. 2). The solid curves represent the dependence of $\operatorname{Re}\gamma$ on k, and the dotted curves the dependence of $\operatorname{Im}\gamma$ on k. The diagrams show that the results of calculations coincide with the exact values given in [5].

Figure 1 shows that in the case at hand the propagation constants of waves of the lower types may be either pure imaginary or pure real, and both waves are normal in the entire frequency range.

Figure 2 shows that in the given case both normal and anomalous waves exist. The values of γ may be complex, i. e., may have real and complex parts.

REFERENCES

1. A.N. Bogolyubov and A.L. Delitsyn *Vestn. Mosk. Univ. Fiz. Astron.*, no. 1, p. 9, 1996 (*Moscow University Phys. Bull.*, no. 1, p. 7, 1996).
2. A.A. Samarskii, *Dokl. Akad. Nauk SSSR*, no. 1, p. 631, 1948.
3. A.N. Bogolyubov, A.L. Delitsyn, and M.D. Malykh, *Zh. Vychisl. Mat. Mat. Fiz,*, vol. 41, no. 1, p. 126, 2001.
4. P.J. Siarlit and J.L. Lions, *Handbook on Numerical Analysis*, vol. 2: *Finite Element Method*, Amsterdam, 1991.
5. G.I. Veselovskii and S.B. Raevskii, *Layered Metal–Insulator Waveguides* (in Russian), Moscow, 1988.

11 October 2004

Department of Mathematics