Moscow University Physics Bulletin
Vol. 60, No. 5, pp. 22–27, 2005

Vestnik Moskovskogo Universiteta. Fizika
UDC 537.52

RADIOPHYSICS

TWO-DIMENSIONAL PLASMA AND SPACE-CHARGE SHEATH EQUATION FOR THE GAS DISCHARGE POSITIVE COLUMN

S. A. Dvinin and A. A. Kuzovnikov

E-mail: dvinin@ph-elec.phys.msu.su

A plasma and space-charge sheath equation that extends the Langmuir–Tonks equation to a two-dimensionally nonuniform plasma has been obtained. This equation can serve as a basis for calculating the spatial characteristics of the plasma of a gas-discharge positive column of arbitrarily shaped boundary under conditions of free flight of ions. The results obtained are of interest in modeling the processes occurring in ion sources and low-pressure plasma chemical reactors.

The positive column of a gas discharge in conditions of free flight of ions was first analyzed by Langmuir and Tonks in 1929 ([1]; see also review [2]). The plasma and space-charge sheath equation obtained by them allows one to calculate the spatial distributions of the electrostatic potential and density of charged particles, as well as the lifetime of the latter, provided that the plasma is unidimensonally nonuniform. At the same time, working chambers of plasma reactors or ion sources are usually of cylindrical (or more complex) shape, the chamber radius and height being approximately equal, so that the plasma unidimensional nonuniformity condition is not satisfied.

In this work, we will obtain a most simple plasma and space-charge sheath equation describing a two-dimensionally nonuniform plasma column in conditions of free flight of ions. Since we do not consider the mechanisms of energy transfer to electrons, the results obtained can be applied to both direct-current discharges and low-pressure high-frequency and microwave discharges.

1. ORIGINAL ASSUMPTIONS AND THE GENERAL FORM OF THE PLASMA AND SPACE-CHARGE SHEATH EQUATION

We consider the positive column of a gas discharge in a spatial region limited by the surface Q. The original assumptions of the positive column model are the same as those for the unidimensional model.

1. The positive column consists of neutral particles, electrons, and ions of one and the same species.

2. The energy distribution of the electrons corresponds to a Maxwell distribution with a temperature T_e independent of the coordinates. The electrons are in equilibrium with the electrostatic field.

3. The ionization mechanism is direct ionization of the neutral particles by electron impact from their ground state. The recombination mechanism is surface recombination on the wall located on the specified surface Q. By virtue of assumption 2, the ionization frequency denoted as ν_i is coordinate-independent.

4. The ions originate with a zero energy and reach the walls in free flight.

5. We also assume that a point exists where the electrostatic potential is at its maximum and that the potential distribution along the path of each individual ion is monotonic. The fulfillment of the latter condition should be verified upon completion of the solution.

Following Langmuir and Tonks, we will proceed from the Poisson equation

$$\Delta\varphi = -4\pi e(n - n_e). \tag{1}$$

Here and elsewhere φ is the ambipolar potential; n_e and n are the densities of the electrons and ions, respectively; and $e > 0$ is the elementary electric charge. Since the electrons are in equilibrium with the electric field, it follows that

$$n_e = n_0 \exp(e\varphi/kT_e), \tag{2}$$

where n_0 is the electron density in the center of the positive column and k is the Boltzmann constant.

To close the system of equations (1)–(2), it is necessary to compute the ion density at the point r. To solve this problem, we will proceed from the balance relations similar to those used in [1–3] when solving unidimensional problems. Let $\mathbf{r}_S(t)$ be the coordinates of the point of origin of an ion that at the instant t after origination finds itself at the observation point r_0*. The set of all the points $\mathbf{r}_S(t)$ is a parametrically specified curve. In a symmetrical coordinate system, the curve $\mathbf{r}_S(t)$ starts at the origin of coordinates (where the potential is at its maximum, $t = \infty$) and terminates at the observation point ($t = 0$). The trajectory $\mathbf{l}(t, t_1)$ of the ion born at the point $\mathbf{r}_S(t)$ at the instant $t_1 = t$ passes through the point r_0. Thus, the curve $\mathbf{r}_S(t)$ and the trajectory of the ion have two common points (Fig. 1). We introduce in the neighborhood of the curves $\mathbf{r}_S(t)$ and $\mathbf{l}(t, t_1)$ the coordinate system (u, v, w) with the base vectors $(\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w)$† such that the conditions $(\boldsymbol{\tau}\mathbf{e}_u) \neq 0$ and $(\boldsymbol{\eta}\mathbf{e}_u) \neq 0$ are satisfied, where $\boldsymbol{\tau} = d\mathbf{r}_S/dt$, $\boldsymbol{\eta} = \partial\mathbf{l}/\partial t_1$ are the tangent vectors to the curves $\mathbf{r}_S(t)$ and $\mathbf{l}(t, t_1)$. In that case, the coordinate u can be used to parametrically specify both of the curves, which we will designate as $\mathbf{r}_S(u)$ and $\mathbf{l}(u_S, u)$‡.

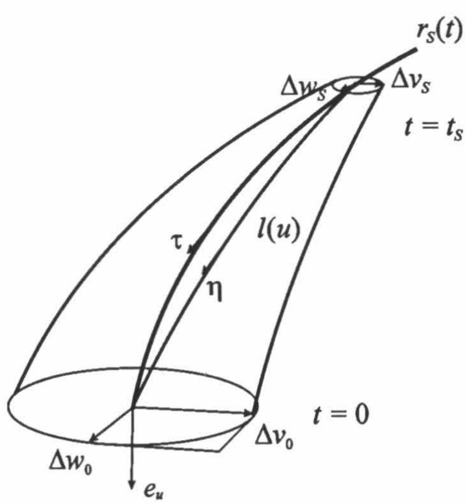

Fig. 1

Mutual arrangement of the ion trajectories l and the set of the points of origin of the trajectories r_S hitting the observation point r_0.

* Hereinafter all the quantities relating to the point of origin of the ion will be marked by the subscript S, whereas those relating to the observation point, by the subscript 0.

† If the coordinate system is specified by the function $\mathbf{r}(u, v, w)$, then $\mathbf{e}_u = \partial\mathbf{r}/\partial u$, $\mathbf{e}_v = \partial\mathbf{r}/\partial v$, $\mathbf{e}_w = \partial\mathbf{r}/\partial w$.

‡ Specifically, either of the variables t and t_1 can be used as the coordinate u.

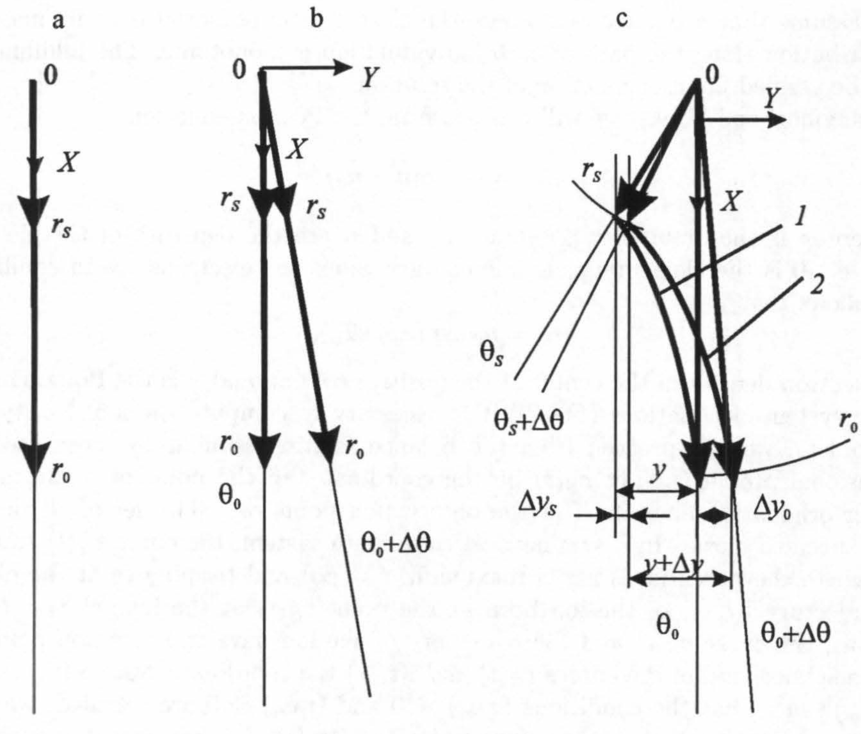

Fig. 2

Mutual arrangement of adjacent ion trajectories: (*a*) plane geometry, (*b*) cylindrical geometry, and (*c*) arbitrary geometry.

We consider a flux of ions born in the neighborhood $(\Delta u_S, \Delta v_S, \Delta w_S)$ of the point $\mathbf{r}_S(u)$ and flowing through the neighborhood $(\Delta v_0, \Delta w_0)^*$ of the observation point in unit time. The number of ions in the flux $\Delta N = \Delta n(\mathbf{V}(u_S, u_S)[\mathbf{e}_{u0} \times \mathbf{e}_{v0}])\Delta v_0 \Delta w_0$, where Δn is the density of these ions in the neighborhood of the observation point. Alternatively, $\Delta N = \nu_i n_e(u_S)(\mathbf{e}_{uS}[\mathbf{e}_{vS} \times \mathbf{e}_{wS}])\Delta u_S \Delta v_S \Delta w_S$. Equating both of the expressions for ΔN, we obtain

$$\Delta n(u_S) = \frac{n_e(u_S)\nu_i(\mathbf{e}_{uS}[\mathbf{e}_{vS} \times \mathbf{e}_{wS}])\Delta u_S \Delta v_S \Delta w_S}{(\mathbf{V}(u_S, u_S)[\mathbf{e}_{u0} \times \mathbf{e}_{v0}])\Delta v_0 \Delta w_0}.$$

Letting Δu_S, Δv_S, and Δw_S tend to zero, considering the functional relationship between $(\Delta v_0, \Delta w_0)$ and $(\Delta v_S, \Delta w_S)$, and integrating with respect to du_S, we get

$$n(u_0) = \int\limits_{\mathbf{r}_S} du_S \frac{n_e(u_S)\nu_i(\mathbf{e}_{uS}[\mathbf{e}_{vS} \times \mathbf{e}_{wS}])}{(\mathbf{V}(u_S, u_0)[\mathbf{e}_{u0} \times \mathbf{e}_{v0}])} \left(\frac{\partial(v_0, w_0)}{\partial(v_S, w_S)} \right)^{-1}. \tag{3}$$

The velocity $\mathbf{V}(u_S, u_0)$ at the observation point u_0 of an ion born at the point u_S, the functional relationship $(\partial(v_0, w_0)/\partial(v_S, w_S))$ of the coordinates, and the integration curve $\mathbf{r}_S(u_S)$ should be found from the equations of motion of the ions in the field of the potential φ. Substituting expressions (2) and (3) into Poisson equation (10) and omitting the subscript 0 of the quantities relating to the observation point, we obtain the following multidimensional plasma and space-charge sheath equation:

$$\Delta\varphi = 4\pi n_0 e \left(\exp\left(\frac{e\varphi(\mathbf{r})}{kT_e}\right) - \nu_i \int\limits_{\mathbf{r}_S} du_S \frac{(\mathbf{e}_{uS}[\mathbf{e}_{vS} \times \mathbf{e}_{wS}])}{(\mathbf{V}(u_S, u_0)[\mathbf{e}_{u0} \times \mathbf{e}_{v0}])} \left(\frac{\partial(v_0, w_0)}{\partial(v_S, w_S)} \right)^{-1} \exp\left(\frac{e\varphi(\mathbf{r}_S)}{kT_e}\right) \right).$$

* u_0, $v_0 + \Delta v_0$, and $w_0 + \Delta w_0$ are the coordinates of a point in the neighborhood of the observation point u_0, through which there passes the trajectory of the ions born at the point $(u_S, \nu_S + \Delta v_S, w_S + \Delta w_S)$, i.e., they are the functions of $(u_S, \Delta v_S, \Delta w_S)$.

On introducing the dimensionless potential $\eta = e\varphi/kT_e$ and designations $r_{De} = \sqrt{kT_e}/4\pi e^2 n_0$, $V_S = \sqrt{kT_e}/M$, where M is the mass of the ions, the sought-for equation will be simplified:

$$r_{De}^2 \Delta\eta = \exp(\eta(\mathbf{r})) - \nu_i \int\limits_{\mathbf{r}_S} du_S \frac{(\mathbf{e}_{uS}[\mathbf{e}_{vS} \times \mathbf{e}_{wS}])}{(\mathbf{V}(u_S, u_0)[\mathbf{e}_{u0} \times \mathbf{e}_{v0}])} \left(\frac{\partial(v_0, w_0)}{\partial(v_S, w_S)}\right)^{-1} \exp(\eta(u_S)). \tag{4}$$

The result obtained by Langmuir and Tonks can easily be derived from equation (4) upon allowing for the functional relationship $\Delta y = (x/x_S)^\mu \Delta y_S$ and the straightness of the lines $\mathbf{r}_S(t)$ and $\mathbf{l}(t, t_1)$ and introducing a rectangular coordinate system in their neighborhood (Fig. 2*a*, *b*). Here $\mu = 0$ for plane geometry, $\mu = 1$ for cylindrical geometry, and $\mu = 2$ for spherical geometry. It follows from equation (4) that

$$r_{De}^2 \Delta\eta = \exp(\eta(\mathbf{r})) - \frac{\nu_i}{\sqrt{2}V_S} \int\limits_0^L \frac{x_S^\mu}{x^\mu} \frac{du_S}{\sqrt{\eta(u_S) - \eta(u)}} \exp(\eta(u_S)). \tag{5}$$

As distinct from the unidimensional problem, where the resultant equation is mathematically exact (within the framework of assumptions (1)–(5) made), when writing down the two-dimensional equation, one has to use, at a certain stage, the perturbation theory. In essence, it is necessary to indicate the explicit form of the curve $\mathbf{r}_S(u)$ and ion trajectories. The position of a point on a trajectory will be a functional of the potential, i.e., it will depend on all the derivatives of the potential in the neighborhood of the curve. When constructing an approximate solution in the simplest model, we will take account of the lower derivatives, using as a small parameter the ratio between length of the trajectory and its radius of curvature. The direction of motion of the ion depends on the ratio E_y/E_x. Curvature arises when this ratio starts depending on the coordinate. We note that curvature as such affects only the shape of the ion trajectory. The ionization balance, hence the potential distribution along the trajectory depends on the rate of divergence of the adjacent trajectories, governed by the spatial potential distribution. Thus, the small parameter will be the quantity $\mu = \left(L \frac{d}{dl} \ln \frac{E_y}{E_x}\right)^{-1}$, where L is the length of the trajectory. The very expression for the small parameter bears witness to the complex form of even the lowest-order approximations. Since different versions of the perturbation theory construction are possible, different forms of the plasma and space-charge sheath equation can exist.

The boundary conditions on the surface Q for equation (5) may be formulated as follows*. For the positive column restricted by a dielectric, the boundary condition has the meaning of the local equality of the electron and ion currents [1–4] at each point q of the boundary surface. To calculate the electron current to the wall, we can use the equation of the standard probe current theory [8] (A is a constant of the order of unity that depends, for example, on the form of the electron energy distribution function and m is the electron mass)

$$\begin{aligned} G(q) = \Bigg\{ & A|(\mathbf{e}_{uq}[\mathbf{e}_{vq} \times \mathbf{e}_{wq}])| \sqrt{\frac{kT_e}{m}} \exp(\eta(\mathbf{r})) \\ & - \nu_i \int\limits_{\mathbf{r}_S} (\mathbf{e}_{uS}[\mathbf{e}_{vS} \times \mathbf{e}_{wS}]) \exp(\eta(u_S)) \left(\frac{\partial(v_q, w_q)}{\partial(v_S, w_S)}\right)^{-1} du_S \Bigg\}\Bigg|_{\mathbf{r}=q} = 0. \end{aligned} \tag{6}$$

In the case of conductive boundary, the equality condition for the electron and ion currents should be imposed on the surface as a whole,

$$\oint\limits_Q G(q)\, dq = 0. \tag{7}$$

And finally in the case of quasineutral approximation ($r_{De} \ll L$), neglecting the processes occurring in the space-charge sheath when $r_{De} = 0$ is put in expression (7), the boundary condition has the meaning of violation of the mutually unambiguous coordinate transformation relation [5]

$$\frac{\partial(x, y, z)}{\partial(\eta, v, w)} = 0. \tag{8}$$

* The points belonging to the boundary surface will be denoted by the subscript q.

2. APPROXIMATE VERSION OF THE PLASMA AND SPACE-CHARGE SHEATH EQUATION

In this section, we will obtain one of the simplest versions of the plasma and space-charge sheath equation, restricting our consideration to the two-dimensional problem. We simplify equation (4) by introducing a cylindrical coordinate system (Fig. 2c). Here u is a coordinate parametrically specified on the curve $\mathbf{r}_S(u)$; r_0, θ_0 are the coordinates of the observation point; $r_S(u, r_0, \theta_0)$, $\theta_S(u, r_0, \theta_0)$ are the coordinates of a point on the curve $\mathbf{r}_S(u)$, at which an ion is born; $V(u, r_0, \theta_0)$ is the velocity that the ion born at this point has when it reaches the observation point; and θ_V is the angle determining the ion velocity direction. When calculating the trajectory of the ion, we will also use a rectangular system of coordinates xy, whose x-axis is directed along the radius (see Fig. 2c).

In the simple model described below, the motion of the ion along the x-axis will be so described as if there is no motion along the y-axis. What is more, we will restrict ourselves to a first-order approximation of the perturbation theory. In that case, $\eta(r_S, \theta_S) = \eta(r_S, \theta_0) - \frac{\partial \eta}{r_S \partial \theta}\Delta y + \ldots$, where $(-\Delta y)$ is the y-coordinate of the point (r_S, θ_S) in the xy coordinate system (see Fig. 1),

$$\begin{aligned} n_e(r_S, \theta_S) &= n_e(r_S, \theta_0)\left(1 - \frac{\partial \eta}{r_S \partial \theta}\Delta y + \ldots\right), \\ V_r = V(u, u_0)\cos(\theta_V(u_0, r_0, \theta_0) - \theta_0) &= V_S\sqrt{2(\eta(r_S, \theta_0) - \eta(r_0, \theta_0))}. \end{aligned} \tag{9}$$

To calculate Δy, we use the Newton equation for ions. Assuming

$$\ddot{y} = -V_S^2\left(\frac{\partial \eta}{\partial y} + \frac{\partial^2 \eta}{\partial y^2}y + \frac{\partial^3 \eta}{2\partial y^3}y^2 + \ldots\right), \tag{10}$$

where all the derivatives are calculated on the ray $\theta = \theta_0 = \text{const}$ and are functions of r and θ_0, we obtain

$$\begin{aligned} \dot{y}(r_S, \theta_S, r) &= -V_S^2 \int_0^t dt' \frac{\partial \eta}{\partial y} = -V_S \int_{r_S}^{r} \frac{dr'}{\sqrt{2(\eta(r_S, \theta_0) - \eta(r', \theta_0))}} \frac{\partial \eta}{\partial y}, \\ y(r_S, \theta_S, r) &= \int_0^t \dot{y}\, dt' - y_S = \int_r^{r_0} \frac{dr'}{2\sqrt{\eta(r_S, \theta_0) - \eta(r', \theta_0)}} \int_{r_S}^{r'} \frac{dr''}{\sqrt{\eta(r_S, \theta_0) - \eta(r'', \theta_0)}} \frac{\partial \eta}{\partial y}. \end{aligned} \tag{11}$$

Comparing the trajectories of the ions born at two adjacent points θ_0 and $\theta_0 + \Delta\theta$ (see Fig. 2c), we get

$$\Delta y(r_S, \theta_0, r) = \frac{r}{r_0}\Delta y_0 + \int_r^{r_0} \frac{dr'}{2\sqrt{\eta(r_S, \theta_0) - \eta(r', \theta_0)}} \int_{r_S}^{r'} \frac{dr''}{\sqrt{\eta(r_S, \theta_0) - \eta(r'', \theta_0)}} \frac{\partial^2 \eta}{\partial y^2}\Delta y(r_S, \theta_0, r''). \tag{12}$$

Relation (12) is an integral equation in Δy. In the limit as $\Delta y \to 0$, equation (12) yields the following equation for the rate of divergence of the ion trajectories:

$$\frac{dy(r_S, \theta_0, r)}{dy_0} = \frac{r}{r_0} + \int_r^{r_0} \frac{dr'}{2\sqrt{\eta(r_S, \theta_0) - \eta(r', \theta_0)}} \int_{r_S}^{r'} \frac{dr''}{\sqrt{\eta(r_S, \theta_0) - \eta(r'', \theta_0)}} \frac{\partial^2 \eta}{\partial y^2} \frac{dy(r_S, \theta_0, r'')}{dy_0}. \tag{13}$$

The first term here takes account of the rotation of the coordinate system due to the variation of the angle θ_0. Considering the relationship between the coordinates x, y and r, θ and solving equation (13) by the perturbation theory method, we get

$$\frac{dy_S}{dy_0} = \left(\frac{r_S}{r_0} + \int_{r_S}^{r_0} \frac{dr'}{2\sqrt{\eta(r_S, \theta_0) - \eta(r', \theta_0)}} \int_{r_S}^{r'} \frac{dr''}{\sqrt{\eta(r_S, \theta_0) - \eta(r'', \theta_0)}} \frac{\partial^2 \eta}{r'' r_0 \partial \theta^2}\right).$$

The above relation, when used in conjunction with formula (9), enables one to calculate the density of ions at the observation point, (3)

$$n = \frac{\nu_i}{\sqrt{2}V_S}\int_0^{r_0} e^{\eta(r_S,\theta_0)}\frac{r_S}{r_0}\frac{dr_S}{\sqrt{\eta(r_S,\theta_0)-\eta(r_0,\theta_0)}}$$
$$\times\left(1+\int_{r_S}^{r_0}\frac{dr'}{2\sqrt{\eta(r_S,\theta_0)-\eta(r',\theta_0)}}\int_{r_S}^{r'}\frac{dr''}{\sqrt{\eta(r_S,\theta_0)-\eta(r'',\theta_0)}}\left(\frac{\partial\eta}{r_S\partial\theta}\frac{\partial\eta}{r''\partial\theta}+\frac{\partial^2\eta}{r_S r''\partial\theta^2}\right)\right),$$

and the plasma and space-charge sheath equation has the form (we have omitted the subscript 0 for the observation point)

$$\exp(-\eta(\mathbf{r}))r_{De}^2\Delta\eta = 1-\frac{\nu_i}{\sqrt{2}V_S}\int_0^{r} e^{\eta(r_S,\theta)-\eta(r,\theta)}\frac{r_S}{r}\frac{dr_S}{\sqrt{(\eta(r_S,\theta)-\eta(r,\theta))}}$$
$$\times\left(1+\int_{r_S}^{r}\frac{dr'}{2\sqrt{\eta(r_S,\theta)-\eta(r',\theta)}}\int_{r_S}^{r'}\frac{dr''}{\sqrt{\eta(r_S,\theta)-\eta(r'',\theta)}}\left(\frac{\partial\eta}{r_S\partial\theta}\frac{\partial\eta}{r''\partial\theta}+\frac{\partial^2\eta}{r_S r''\partial\theta^2}\right)\right). \quad (14)$$

If the azimuthal derivatives are equal to zero, equation (14) implies the result obtained by Langmuir and Tonks, (5) [1]. The additional terms in (14) describe the azimuthal equalization of the potential and the growth of the density of ions as a result of their azimuthal drift (the ions come from a region of higher electron density). Both of these effects prove to be of nonlocal character, for they depend on the properties of the potential at all the points visited by the ion. The ionization frequency is the eigenvalue of equation (14). In those trajectories where the distance $R(\theta_0)$ to the boundary is minimal, the azimuthal field reduces the divergence of the trajectories, whereas near the maximum of $R(\theta_0)$, it, on the contrary, increases the trajectory divergence, owing to which the particle balance condition is satisfied in each of the trajectories at one and the same ionization frequency. The methods of solving equations similar to equation (14) will be considered in the subsequent works.

CONCLUSIONS

In this work, we have derived an integro-differential equation describing the potential distribution in a two-dimensionally nonuniform plasma, similar to the plasma and space-charge sheath equation obtained by Langmuir and Tonks for the cylindrical positive column of a gas discharge. As distinct from the unidimensional problem, the equation obtained by us proves to be approximate, the small parameter used is the ratio between the characteristic size of the plasma column and the radius of curvature of the ion trajectory. The results presented in the paper are of interest for the analysis of the processes occurring in low-pressure plasma chemical reactors.

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19 July 2004 Department of Physical Electronics