

BEAM INSTABILITY IN A SINGLE-PARTICLE CHERENKOV EFFECT NEAR THE OPACITY BAND

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Instability associated with stimulated emission of radiation by an electron beam via the single-particle Cherenkov effect in an electrodynamic system near the opacity band is examined. Such instability is shown to be absolute. The instability increments and wave number ranges where instability occurs are determined, and the complex-valued dispersion dependences are investigated.

The study of the stability of a system subjected to small monochromatic perturbations amounts to solving the dispersion equation $D(\omega, k) = 0$, which is derived as a result of linearizing the “equations of motion” of the system with respect to the small perturbations $\sim \exp\{-i\omega t + ikx\}$. The frequency $\omega = \omega(k)$ determines the temporal dynamics of the development of perturbations with the wave number k . If the imaginary part of the frequency is positive, $\text{Im}\omega > 0$, in a certain range of values of k , the perturbations build up exponentially with time, and the system is unstable. However, the sign of $\text{Im}\omega$ alone does not suggest that the perturbations increase at a fixed point in space.

Actually, a perturbation is not a monochromatic wave but a set of such waves, or a wave packet. The monochromatic waves comprising the packet exponentially grow in the instability range, while the packet as a whole drifts in the process. It is the competition of these two opposite factors, the increase of the amplitude of the perturbations caused by the instability and the decrease caused by the displacement of a packet limited in space, that determines the behavior of the perturbations at a fixed point x in space as $t \rightarrow +\infty$.

To adequately describe the dynamics of the development of perturbations in an unstable system needs, generally speaking, solving the initial-value problem. However, it can be shown that as $t \rightarrow +\infty$, the nature of the instability is exclusively determined by the dispersion equation and is independent of the form of the initial conditions [1–4]. If the perturbations at a fixed point of space of an unstable system grow with time, we usually speak of absolute instability. But if these perturbations do not increase at a fixed point x (usually they tend to zero), then we speak of convective (drift) instability. The nature of the instability is determined by the branch points of the functions $k(\omega)$, the solutions of the dispersion equation. If there is at least one branch point $\omega = \omega_1 + i\omega_2$ with a positive imaginary part, $\omega_2 > 0$, and this is the branch point of waves $k = k(\omega)$ corresponding to different directions of propagation, we have absolute instability. If there are no such branch points in the functions $k(\omega)$, the instability is convective.

Let us examine the dispersion equation

$$(\omega - \omega_p(k)) ((\omega - ku)^2 - \omega_b^2(k)) = \theta, \quad (1)$$

which describes, e. g., the instability of a beam of electrons moving with a velocity u in a plasma waveguide or some other electrodynamic system [5, 6]. The functions $\omega_p(k)$ and $\omega_b(k)$ determine the spectra of the frequencies of the electrodynamic system and the beam waves, respectively. The parameter θ stipulates the coupling of these waves, and it may be either positive or negative. In the case of low-density beams,

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i. e., $\omega_b(k) \ll ku$, we speak of a single-particle Cherenkov effect, with the beam waves being degenerate and system (1) becoming much simpler,

$$(\omega - \omega_p(k))(\omega - ku)^2 = \theta. \quad (2)$$

The strongest coupling of the waves occurs near the point where their dispersion curves intersect. This point can be found by solving the following system of equations:

$$\omega_0 = \omega_p(k_0), \quad \omega_0 = k_0 u. \quad (3)$$

Assuming that the wave packet is sufficiently narrow, we expand the function $\omega_p(k)$ in a series in $k - k_0$. Limiting ourselves to the linear approximation, we find that

$$(\omega - \omega_0 - v_p(k - k_0))(\omega - \omega_0 - u(k - k_0))^2 = \theta, \quad (4)$$

where $v_p = \frac{\partial \omega_p}{\partial k}(k_0)$ is the group velocity of the wave of the electrodynamic system.

The nature of the instability of a system with dispersion equation (2) is determined by two rules (irrespective of the sign of the parameter θ) [7]:

- (1) if $v_p u > 0$, instability is convective and there is a buildup of vibrations in the system; and
- (2) if $v_p u < 0$, instability is absolute.

We note that in the second case the dispersion equation (2) has complex-conjugate solutions with respect to the wave number $k(\omega)$. Nevertheless, in this case we should not speak of vibration buildup. The thing is that the solutions $\sim \exp\{-i\omega t + ik(\omega)x\}$ describe the steady-state behavior of the system as $t \rightarrow +\infty$ (after all the transient processes related to the switch-on of perturbations at $x = 0$ have terminated). In the event of absolute instability, it is simply impossible for steady state to set in.

Near the opacity band of the system, $v_p = 0$ and we must take into account the following term in the expansion in $k - k_0$:

$$(\omega - \omega_0 - \alpha(k - k_0)^2)(\omega - \omega_0 - u(k - k_0))^2 = \theta. \quad (5)$$

Introducing new dimensionless variables, instead of (5) we get

$$(\tilde{\omega} - \tilde{\alpha}\tilde{k}^2)(\tilde{\omega} - \tilde{k})^2 = \tilde{\theta}, \quad (6)$$

where $\tilde{\theta} = \pm 1$. This dispersion equation describes the excitation by an electron beam of electrodynamic systems of vacuum microwave electronics near the opacity band [8-10]. In plasma microwave electronics, the extremum of the dispersion curve belongs to the cyclotron branch in a finite magnetic field. In what follows we drop the tilde, thus retaining the old notation, and assume that $\alpha > 0$.

The authors of [11] examined an equation similar to (6) but ω is ignored (in comparison to k) in the second factor on the left-hand side. However, ignoring ω actually results in an equation that describes stable vibrations and also distorts the asymptotic behavior of the two roots of $k(\omega)$ as $\text{Im } \omega \rightarrow \infty$, which is important if we want to classify waves by their directions of propagation. In the present work we analyze the dispersion equation with the exact frequency dependence.

Even without explicitly writing the solution $\omega(k)$ of (6), we easily establish that a system described by this equation is unstable (has complex-conjugate roots) for all values of the parameters. Here the range of wave numbers k in which instability develops is determined by the inequality

$$\frac{4}{27}\theta(k - \alpha k^2)^3 < 1. \quad (7)$$

If $\theta = +1$ and $\alpha > 1/6\sqrt[3]{2}$, the system is unstable for all values of k . The behavior of the dispersion curves in this case is shown in Fig. 1a ($\alpha = 1$). Figure 1b depicts the behavior of the positive imaginary part of the complex-valued frequency ω (increment).

For $\alpha < 1/6\sqrt[3]{2}$, there are two semi-infinite regions of k in which instability develops:

$$k < \frac{1 - \sqrt{1 - 6\sqrt[3]{2}}}{2\alpha}, \quad k > \frac{1 + \sqrt{1 - 6\sqrt[3]{2}}}{2\alpha}. \quad (8)$$

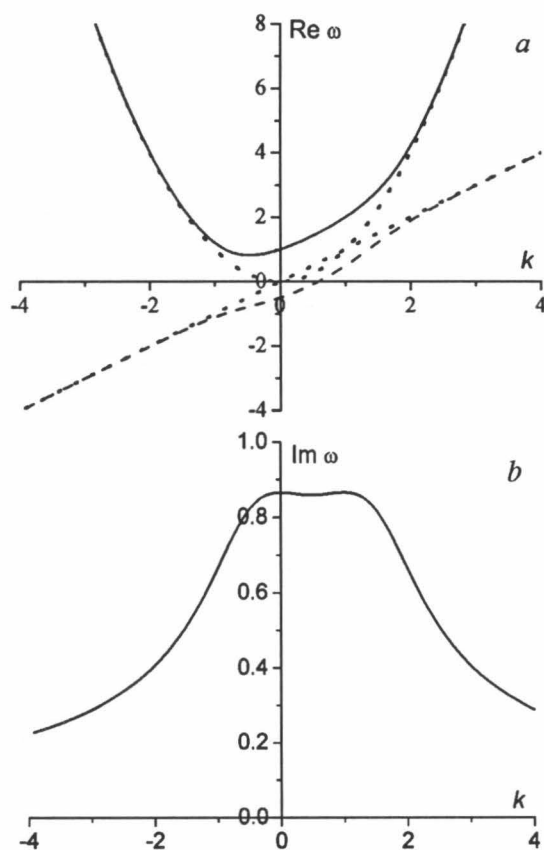


Fig. 1

(a) Dispersion curves and (b) instability increment at $\theta = +1$, $\alpha = 1$. The dashed curve represents $\text{Re } \omega$ in the instability range, and dotted curves represent the same waves in the absence of coupling.

The shape of the dispersion curves and the instability increment as a function of k at $\alpha = 0.1$ are shown in Fig. 2a, b. For large values of k the increment tends to zero, just as it does in the first case.

At $\theta = -1$ and in the range of wave numbers determined by the inequalities

$$\frac{1 - \sqrt{1 - 6\sqrt[3]{2}}}{2\alpha} < k < \frac{1 + \sqrt{1 - 6\sqrt[3]{2}}}{2\alpha}, \quad (9)$$

the frequencies ω are complex-conjugate, and instability develops in the system. Figure 3 shows the dispersion curves (Fig. 3a) and instability increment (Fig. 3b) at $\alpha = 1$.

Let us now determine the nature of the instability of a system described by the dispersion equation (6). The branch points of the functions $k = k(\omega)$, the solutions of the dispersion equation (6), are determined by the following system of equations [1-4]:

$$D(\omega, k) = 0, \quad \frac{\partial D(\omega, k)}{\partial k} = 0. \quad (10)$$

By combining dispersion equation (6) and (10), we arrive at a seventh-order equation with respect to ω for finding the branch points $\omega = \omega_1 + i\omega_2$. Depending on the values of the parameters of the system, we have either two or three branch points with positive imaginary parts, $\omega_2 > 0$.

For a given value of ω , dispersion equation (6) determines four waves with different k 's.

Tending the imaginary part of the frequency to infinity while the real part remains fixed, we obtain expressions for the asymptotic behavior of the roots,

$$k_{1,2}(\omega) \cong i \text{Im } \omega, \quad k_{3,4}(\omega) \cong \pm(1 + i)\sqrt{\text{Im } \omega/2\alpha}. \quad (11)$$

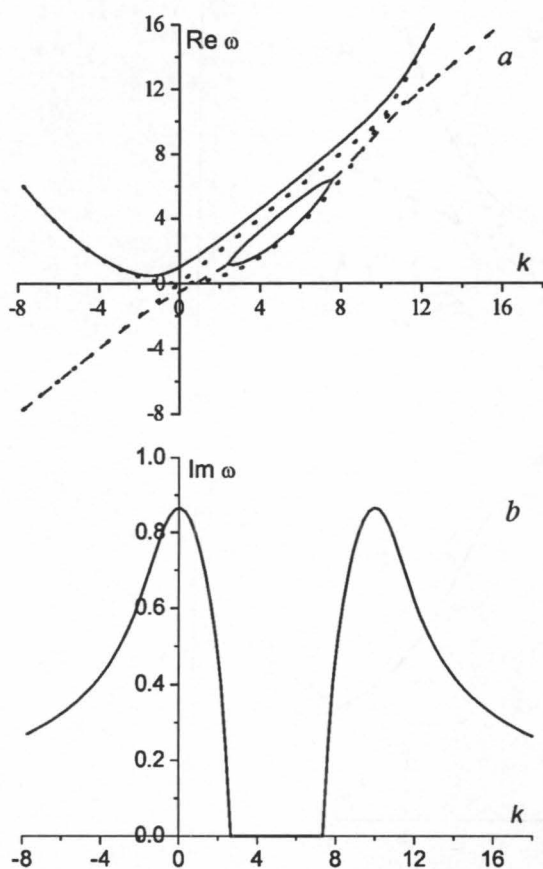


Fig. 2

(a) Dispersion curves and (b) instability increment at $\theta = +1$, $\alpha = 0.1$. The dashed curve represents $\text{Re}\omega$ in the instability range, and dotted curves represent the same waves in the absence of coupling.

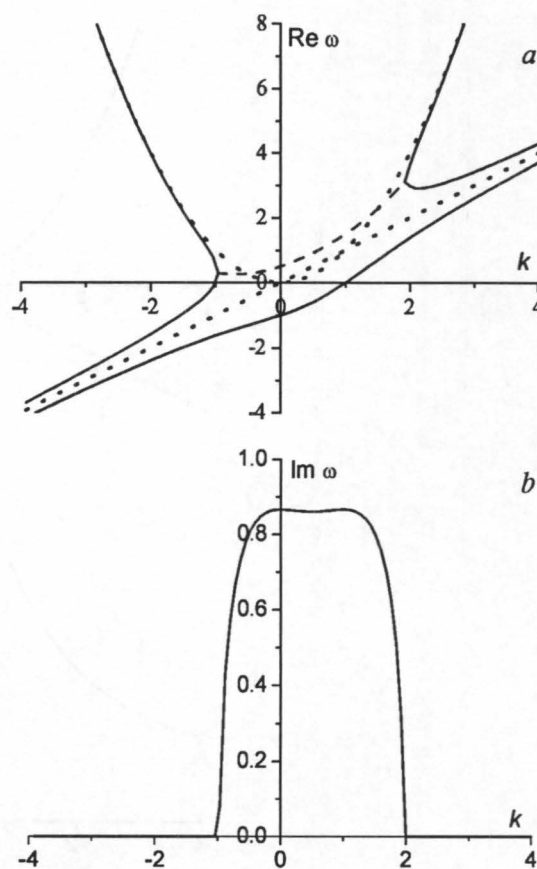


Fig. 3

(a) Dispersion curves and (b) instability increment at $\theta = -1$, $\alpha = 1$. The dashed curve represents $\text{Re}\omega$ in the instability range, and dotted curves represent the same waves in the absence of coupling.

Three roots out of the four correspond to waves that decay in the positive direction of the x -axis, while the fourth root corresponds to a wave decaying in the negative direction of the x -axis. The limit $\text{Im}\omega \rightarrow +\infty$ means an instant switch-on of a source of perturbations with frequency $\text{Re}\omega$ at a point x in space, which implies that the waves decaying in the given direction, as $\text{Im}\omega \rightarrow +\infty$, should be considered waves that propagate in that direction with frequency $\text{Re}\omega$ [1-4]. Such classification of waves by their direction of propagation agrees with the causality principle. The nature of the instability of the system is determined by the behavior of the roots of $k(\omega)$ near the branch points with $\text{Im}\omega > 0$. What we mean is that if there is at least one branch point with $\text{Im}\omega > 0$ at which the roots of $k(\omega)$ corresponding to waves with different directions of propagation branch (the imaginary parts $\text{Im}k$ have different signs, as $\text{Im}\omega \rightarrow +\infty$), then the system is absolutely unstable. Here the asymptotic behavior of the perturbations, as $t \rightarrow +\infty$, is determined by the branch point of waves $k(\omega)$ with different directions of propagation and the largest imaginary part, i. e., $\sim \exp\{\omega_2 t - i\omega_1 t\}$. In all other cases the unstable system is convectively unstable [1-4].

Figure 4 shows the behavior of the roots of $k(\omega)$ in the complex plane for the case where $\alpha = 1$ and $\theta = 1$ as the frequency changes along the contour with $\text{Re}\omega = \omega_1$ and $+\infty > \text{Im}\omega > \omega_2$. The four sets of points with different markings correspond to four different branches of $k(\omega)$. Reducing the imaginary part of the frequency, $\text{Im}\omega$, from a fairly large value to ω_2 while the real part ω_1 remains fixed, we obtain these four sets of points of $k(\omega)$ two of which tend to the same limit (the point $k = -0.4 - 0.45i$ in Fig. 4). Since the roots of $k(\omega)$ corresponding to waves with different directions of propagation converge at the given branch

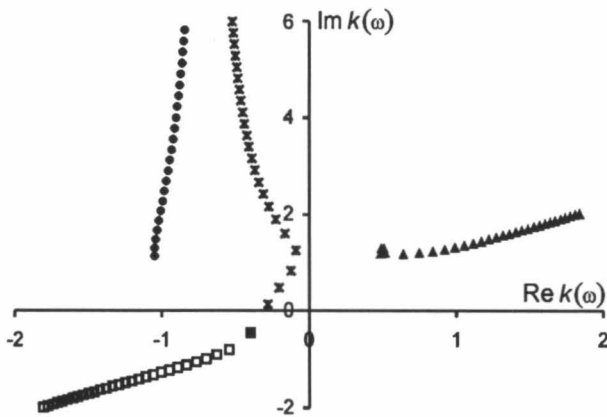


Fig. 4

Behavior of the roots of $k(\omega)$ in the complex plane as the frequency ω changes from $\omega_1 + i\infty$ to $\omega_1 + i\omega_2$.

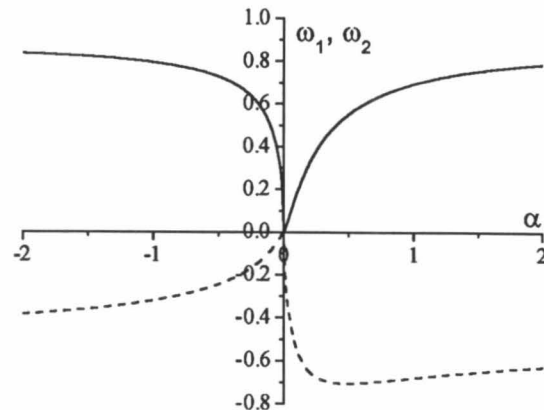


Fig. 5

The real part ω_1 (dashed curve) and the imaginary part ω_2 (solid curve) of the frequency at which absolute instability develops as functions of α .

point, we are dealing with the case of absolute instability [1-4]. Doing similar calculations for other values $\alpha > 0$ and all other branch points of $k(\omega)$ with positive imaginary parts ω_2 , we see that the instability is absolute for all $\alpha > 0$.

The case where $\alpha < 0$ requires no special treatment because the substitutions $\alpha \rightarrow -\alpha$, $k \rightarrow -k$, $\omega \rightarrow -\omega$, and $\theta \rightarrow -\theta$ transform (6) into the same equation but with a positive factor of k^2 and the opposite value of θ . However, the procedure by which the nature of instability is determined has changed. In terms of the new variables, $\text{Im } \omega$ should be varied from $-\infty$ to $-\omega_2 < 0$. But since dispersion equation (6) has real coefficients, changing the sign of $\text{Im } \omega$ leads only to a change in all signs of $\text{Im } k(\omega)$, which means that nature of the instability does not change. Here the perturbations at a fixed point in space will build up as $t \rightarrow +\infty$ with the same increment ω_2 and the opposite value of the frequency ω_1 .

Figure 5 shows the instability increment ω_2 (solid curve) and the frequency ω_1 (dashed curve) of the asymptotic behavior, as $t \rightarrow +\infty$, of the perturbations $\sim \exp\{\omega_2 t - i\omega_1 t\}$ as functions of the parameter α at $\theta = 1$. As noted earlier, a system described by dispersion equation (6) is always absolutely unstable. The case where $\theta = -1$ is obtained via mirror reflection of the solid curve in Fig. 5 with respect to the vertical axis and of the dashed curve, with respect to the origin. For values of α that are not close to zero, ω_2 is of order of the instability increment of a monochromatic wave. As $\alpha \rightarrow 0$, meaningful analysis requires that we allow for the next term in the expansion (proportional to k^3) in the first factor of the left-hand side of (6).

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