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Vestnik Moskovskogo
Universiteta. Fizika
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BRIEF COMMUNICATIONS

THEORETICAL AND MATHEMATICAL PHYSICS

THE THEOREM OF INSTABILITY OF EMBEDDED TRAPPED MODES

A. N. Bogolyubov and M. D. Malykh

E-mail: malykham@mtu-net.ru

A new justification of the theory of perturbations for the waveguide embedded trapped modes is proposed.

We consider the spectral problem

$$\begin{cases} \Delta u + \lambda q u = 0, & x \in X, \\ u = 0, & x \in \partial X, \end{cases} \tag{1}$$

with the condition

$$\|u\|^2 = \int\limits_X u^2 \, d\tau = 1.$$

Here X is a cylinder of constant section S, i. e., a "waveguide", and q is a real piecewise continuous function with compact support $\operatorname{supp}(q-1)$ that characterizes the waveguide filling. From the physical viewpoint, these eigenfunctions are standing waves with frequency $\sqrt{\lambda}$ and with no energy transport. A major part of their energy is concentrated in a finite region, i. e., in a "trap" formed by inhomogeneity of the filling. This is why they are called trapped modes [1].

Let the Ox-axis be directed along the waveguide, and variables that vary in the section S be denoted as y. Then, let the eigenfunctions and eigenvalues of the Dirichlet problem in the section S be denoted as ψ_n and α_n, respectively, i. e.,

$$\Delta\psi + \alpha^2\psi = 0, \quad \psi \in \overset{\circ}{W}{}^1_2(S).$$

The continuous spectrum of problem (1) starts with α_1^2. With the help of the Rayleigh principle, it was demonstrated that problem (1), with $q \geq 1$, should necessarily have at least one eigenvalue smaller than α_1^2, i. e., not embedded into the continuous spectrum. In order to investigate the existence of embedded trapped modes, i. e., eigenvalues $\lambda > \alpha_1^2$, we should raise the lower boundary of the continuous spectrum, so that we could apply the Rayleigh principle and the regular perturbation theory of Rellich–Kato.

Let u be an eigenfunction of problem (1), corresponding to the eigenvalue $\lambda \in (\alpha_r^2, \alpha_{r+1}^2)$, then at $n = 1, \dots, r$, functions

$$u_n(x) := \int\limits_S u(x,y)\psi_n(y)\, dy$$

have a compact support, and for $n > r$,

$$u_n(x) := \int\limits_S u(x,y)\psi_n(y)\, dy \in \overset{\circ}{W}{}^1_2(\mathbb{R}^1).$$

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Hence, u is an eigenfunction of the problem

$$\begin{cases} \Delta u + \lambda q u = 0, & x \in X, \\ u = 0, & x \in \partial X, \\ u_n|_{x=\pm a} = 0 & (n = 1, \ldots, r), \\ u_n \in \overset{\circ}{W}{}_2^1(\mathbb{R}^1) & (n = r+1, \ldots). \end{cases} \quad (2)$$

The continuous spectrum of this problem considered in the Hilbert space $\mathfrak{H}$ of all functions of the form

$$v = v_1(x)\psi_1(y) + \ldots + v_r(x)\psi_r(y) + w,$$

where $v_n \in \overset{\circ}{W}{}_2^1([-a, a])$, $w \in \overset{\circ}{W}{}_2^1(X)$ and

$$\int\limits_S w\psi_n \, dy = 0 \quad (n = 1, \ldots, r),$$

starts with α_{r+1}^2, i.e. u is the eigenfunction of problem (2), corresponding to an isolated eigenvalue. This enables us to give a considerably simpler proof of our theorem on unstable embedded trapped modes with small perturbations of the waveguide filling [2].

As is known, problem (1) with $q = q_0(x) \geq 1$ has a trapped mode of the form $u_0(x, y) = u_0(x)\psi_{r+1}(y)$ that corresponds to the eigenvalue $\lambda = e_0 > \alpha_r^2$. Let us demonstrate that, not for every perturbed filling

$$q = q_0(x) + \varepsilon q_1(x, y),$$

there is an eigenvalue $\lambda = e(\varepsilon)$ tending to e_0 as $\varepsilon \to 0$.

We assume the opposite. Then $e(\varepsilon)$, $u(x, y; \varepsilon)$ are isolated eigenvalue and eigenfunction of problem (2), and hence, in virtue of the fact that the problem is self-conjugate, they can be expanded in series in powers of ε,

$$e(\varepsilon) = e_0 + e_1\varepsilon + \ldots, \quad u = u_0(x)\psi_{r+1}(y) + \varepsilon u_1(x, y) + \ldots .$$

Multiplying (1) by $\psi_1(y)$ and integrating it over the whole section S yields

$$\frac{d^2}{dx^2} \int\limits_S dy \, u(x, y)\psi_1(y) + e \int\limits_S dy \, q(x, y) u(x, y)\psi_1(y) = \alpha_1^2 \int\limits_S dy \, u(x, y)\psi_1(y).$$

Upon substituting series for $e(\varepsilon)$ and $u(\varepsilon)$ in the above equation and denoting

$$\int\limits_S dy \, u(x, y)\psi_1(y) = v(x),$$

we obtain in the first order of perturbation theory

$$\frac{d^2 v}{dx^2} + [e_0 q_0(x) - \alpha_1^2] v = e_0 u_0(x) \int\limits_S dy \, q_1(x, y)\psi_1(y)\psi_2(y).$$

In order that $u(x, y; \varepsilon)$ belong to L^2, it is necessary that $v(x)$ belong to $L^2(\mathbb{R}^1)$ as well. Since the support of the perturbed filling $q(x, y) - 1$ is limited, this equation has a solution that belongs to space L^2, only under rather special conditions put on $q_1(x, y)$. This proves the following

Theorem. *There exist piece-wise real perturbations $q_1(x, y)$ of the initial filling $q_0(x)$, such that in the vicinity of the unperturbed eigenvalue there are no perturbed eigenvalues.*

REFERENCES

1. A.N. Bogolyubov and M.D. Malykh, *Radiotekhn. Elektronika*, vol. 50, no. 2. p. 218, 2005.
2. A.N. Bogolyubov, M.D. Malykh, and A.G. Sveshnikov, *Dokl. Ross. Acad. Nauk*, vol. 385, no. 6, p. 744, 2002.

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