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RENORMALIZATION OF THE CASIMIR ENERGY WITH LOGARITHMIC DIVERGENCES

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A procedure for the renormalization of the vacuum field energy in the case where the energy levels are known only numerically is proposed. It requires neither an explicit transcendental equation for determining the levels nor an analytical expression for the asymptotic behavior of these levels. The procedure yields an unambiguous result even when there is a logarithmic divergence in the expression for the vacuum energy.

INTRODUCTION

In 1948, Casimir [1] found the corrections to the energy of a macroscopic system that are caused by vacuum fluctuations of quantized electromagnetic field. The Casimir effect is an observable quantum effect that plays an important role in both micro- and macrolevels in various areas of physics [2, 3]. However, the calculation of the vacuum energy of a quantized field with nontrivial boundary conditions (we call this the Casimir energy) constitutes, except for the simplest cases of free fields confined in cavities with a flat boundary, a serious problem.

Take the simplest case of a free field within a sphere, i. e., in a system for which there exist explicit transcendental equations for the spectrum. A large number of papers dealing with this system have been written: an electromagnetic field inside a conducting sphere was studied in [4–6]; fermions inside a sphere with the confinement condition, adopted in quark bag models were studied in [7–11]; and effects associated with finite fermion masses were studied in [12, 13]. Nevertheless, the first analytical results in a sufficiently closed form for a massive scalar field were obtained only in [14], while for fermions in [15]. Note that to a large extent these results are based on the existence of an explicit transcendental equation for the spectrum: when explicit equations for the levels exist, a transition from sums containing implicit expressions to integrals containing only explicit expressions is possible [16].

One of the greatest difficulties in renormalizing the Casimir energy is the ambiguity related to logarithmic divergences. Indeed, in the absence of logarithmically divergent terms (for example, in the case of massless fields in regions with a flat boundary), the energy is determined by a single dimensional parameter L, the characteristic linear size. The second dimensional parameter of the problem is the regularization parameter α, which we also assumed to have the same dimensions, length. Hence, when there is no logarithmic divergence, "minimal subtraction," which amounts to dropping the terms in the expression for the energy that are singular in α, is a procedure that is not only natural but has also been thoroughly substantiated. From obvious dimensional considerations it follows that any term proportional to α^{-s} ($s > 0$) depends on L as L^{s-1}, i. e., is proportional to a nonnegative power of L. This allows us to normalize the final result at point $L = \infty$, where the Casimir energy vanishes, and to subtract all singular terms. After subtraction is completed there remains, in the limit $\alpha \to 0$, only the term $c\alpha^0/L$ (with a correct dependence on L), which is the final result.

When there is logarithmic divergence, subtraction becomes ambiguous: to renormalize the term $c\alpha^0 \ln(\alpha/L)/L$, we are forced to subtract $c\alpha^0 \ln(d\alpha/L)/L$ from it (d is an arbitrary constant). This leads to a d-related indeterminate form in the final result, and it is impossible to normalize this result as $L \to \infty$, since the indeterminate form is const$/L$, i. e., it appears in the constant that we must find.

For free fields in a sphere, a logarithmic divergence appears in the massless case as a consequence of the boundary's curvature. Here the ambiguity in the renormalization procedure is avoided by using the following method [3, 9]: to the "interior" problem (only fields inside the cavity are taken into account) we add the "exterior" problem (fields outside the cavity are taken into account). In the case at hand, the logarithmic divergences of the exterior and interior problems cancel out and the renormalization procedure becomes unique.

However, with such an approach we are dealing not with renormalization of the vacuum energy of fields inside the cavity but with renormalization of the vacuum energy of a modified problem, where fields exist inside and outside the cavity.

The goal of our work was to develop a method that would make it possible to find the Casimir energy for problems in which, due to computational difficulties, there can be no analytical expression for the energy levels, i. e., the levels can be found only by employing numerical methods. The proposed procedure also avoids the ambiguity related to logarithmic divergence and makes it possible to isolate the sought finite part in the initial (rather than modified) expression for the vacuum energy.

THE ONE-DIMENSIONAL CASE

The simplest system in which the Casimir energy exhibits a logarithmic divergence is the one-dimensional massive scalar field on a (linear) segment of length L with zero boundary conditions at the segment's end points. The initial expression for the Casimir energy in this case is

$$\mathcal{E}_{\text{Cas}} = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n = \frac{1}{2}\sum_{n=1}^{\infty} \sqrt{(\pi n/L)^2 + m^2}. \tag{1}$$

A straightforward approach [1–3] to the renormalization of this quantity amounts to subtracting from the sum (1) the energy which the field would have on the segment L without being limited to this segment. In view of the trivial form of the counterterms, such a procedure is quite effective in the case of a one-dimensional massive field and leads to the correct result. However, already for a field in a spherical cavity this reasoning is insufficient for consistent renormalization [3, 14]. Hence, we will not compute the difference in the energies of two different systems, i. e., a quantity whose physical meaning is not so obvious, to say the least, but instead will renormalize equation (1) directly.

To regularize the sum (1), we must introduce a parameter α, whose dimensions are those of length, that enters in the cutoff function $F(\alpha\omega_n)$:

$$\mathcal{E}^{(r)}_{\text{Cas}} = \frac{1}{2}\sum_{n=1}^{\infty} \omega_n F(\alpha\omega_n). \tag{2}$$

The asymptotic expression for ω_n at $n \gg 1$ contains the linearly divergent term $\pi n/L$ and the logarithmically divergent term $m^2 L/(2\pi n)$, with the result that the expansion of $\mathcal{E}^{(r)}_{\text{Cas}}$ in a power series in α will have the following structure:

$$\mathcal{E}^{(r)}_{\text{Cas}} = c_2\frac{L}{\alpha^2} + c_1\frac{L^0}{\alpha^1} + c_\lambda m^2 L \ln\frac{L}{\alpha} + c_0\frac{\alpha^0}{L} + \text{terms that vanish as } \alpha \to 0.$$

In the case at hand, any subtraction leads to an indeterminate form, and so we must avoid the subtraction procedure. Instead of calculating $\mathcal{E}^{(r)}_{\text{Cas}}$, we find

$$\partial^2_L \mathcal{E}^{(r)}_{\text{Cas}} \cong c_0\frac{2}{L^3} + c_\lambda\frac{m^2}{L} + \cdots .$$

This expression is regular as $\alpha \to 0$ and is uniquely determined by $\mathcal{E}^{(r)}_{\rm Cas}$, which means it does not depend on the way we separate the latter into the singular and nonsingular parts. Knowing $\partial_L^2 \mathcal{E}^{(r)}_{\rm Cas}$, we can reconstruct the sought vacuum field energy $E^{(r)}_{\rm Cas}$, since from the regularity in α and the uniqueness it immediately follows that

$$\partial_L^2 E^{(r)}_{\rm Cas} = \partial_L^2 \mathcal{E}^{(r)}_{\rm Cas}, \tag{3}$$

while the initial conditions for calculating $E^{(r)}_{\rm Cas}$ follow from obvious physical consideration, i.e., for any meaningful regularization both $E^{(r)}_{\rm Cas}$ and $\partial_L E^{(r)}_{\rm Cas}$ must vanish as $L \to \infty$,

$$E^{(r)}_{\rm Cas}(L \to \infty) = \partial_L E^{(r)}_{\rm Cas}(L \to \infty) = 0.$$

After all this, the Casimir energy of the field can be obtained by finding the limit $E_{\rm Cas} = \lim\limits_{\alpha \to 0} E^{(r)}_{\rm Cas}$.

Now we will formalize the above procedure so that it does not depend on the explicit analytical expression for the spectrum. Thus, we have a certain spectrum ω_n for which the Casimir energy contains a logarithmic divergence. First we find a parameter μ such that

$$\sum_{n=1}^{\infty} \sqrt{\omega_n^2 - \mu^2}\, F(\alpha \sqrt{\omega_n^2 - \mu^2}) \tag{4}$$

does not contain a logarithmic divergence. For a massive one-dimensional field μ is equal to the field's mass m. In a more complicated case, for example, a scalar field in a spherical cavity, this will be a parameter with dimensions of those of mass that characterizes the total coefficient (summed over all angular momenta) of the logarithmic divergence.

Then we "add" to the field the mass $\mathcal{M}$ and examine the dependence of the Casimir energy on this added mass in the entire range from $\mathcal{M} = 0$ to $\mathcal{M} = \infty$. It is convenient to introduce a parameter M such that $M^2 \equiv \mathcal{M}^2 + \mu^2$ and to examine the dependence of the Casimir energy on M,

$$\mathcal{E}_{\rm Cas}(M) = \frac{1}{2} \sum_{n=1}^{\infty} \sqrt{\omega_n^2 + M^2 - \mu^2},$$

in the range from $M = \mu$ to $M = \infty$. To this end we find (numerically, since by definition there can be no analytical expression for the levels) for various values of M the following quantity:

$$\partial_M^2 \left(\frac{\mathcal{E}^{(r)}_{\rm Cas}(M)}{M} \right) = \partial_M^2 \left[\frac{1}{2} \sum_{n=1}^{\infty} \sqrt{\frac{\omega_n^2 - \mu^2}{M^2} + 1}\, F\left(\alpha \sqrt{\frac{\omega_n^2 - \mu^2}{M^2} + 1} \right) \right]. \tag{5}$$

Here, in contrast to (2), the argument of the cutoff function $F(x)$ is a dimensionless quantity, with the result that α is also dimensionless. The transition to $\mathcal{E}^{(r)}_{\rm Cas}(M)/M$ stems from the fact that this quantity becomes regular after double differentiation with respect to M. For $\mathcal{E}^{(r)}_{\rm Cas}(M)$ we would be forced to increase the number of differentiations to three, which is undesirable from the computational viewpoint. It is also important that after such a transition the mass M in the transformed spectrum $\tilde{\omega}_n = \sqrt{(\omega_n^2 - \mu^2)/M^2 + 1}$ acts as an effective "length", unity acts as the effective mass, and at $M = \mu$ the initial spectrum ω_n divided by μ is restored.

Equation (5) is constructed in such a way that all the divergent terms disappear after the differentiation procedure is carried out. Indeed, as $n \to \infty$,

$$\sqrt{\frac{\omega_n^2 - \mu^2}{M^2} + 1} \cong \frac{\sqrt{\omega_n^2 - \mu^2}}{M} + \frac{M}{2\sqrt{\omega_n^2 - \mu^2}} + \cdots,$$

with the result that, according to the definition of μ in terms of (4), the sum generated by the first term contains no logarithmic divergence, while the other divergences are proportional to $(M/\alpha)^2/M$ and $(M/\alpha)^1/M$ and vanish after double differentiation. The second term generates a logarithmic divergence, but the coefficient of this divergence is proportional to M^1, so that this divergence also vanishes after double differentiation.

Thus, expression (5) is regular as $\alpha \to 0$. Then the sought finite vacuum energy $E^{(r)}_{\text{Cas}}(M)$ can be found through a relation similar to (3), precisely,

$$\partial^2_M \left(\frac{E^{(r)}_{\text{Cas}}(M)}{M}\right) = \partial^2_M \left(\frac{\mathcal{E}^{(r)}_{\text{Cas}}(M)}{M}\right), \tag{6}$$

with the adopted normalization point as $M \to \infty$,

$$E^{(r)}_{\text{Cas}}(M \to \infty) = \partial_M E^{(r)}_{\text{Cas}}(M \to \infty) = 0. \tag{7}$$

Indeed, the field with the infinite mass M that we added must have a zero vacuum energy. Hence, using (6) to calculate the second derivative $\partial^2_M E^{(r)}_{\text{Cas}}(M)$ for all $M \in [\mu, \infty]$, we can then restore, employing the initial conditions, the sought quantity $E_{\text{Cas}} = \lim_{\alpha \to 0} E^{(r)}_{\text{Cas}}(M = \mu)$, which corresponds to the initial spectrum.

Implementation of this procedure for a massive one-dimensional scalar field with $m = \mu = 1$ on a segment of length L with zero boundary conditions at the segment's end points yields

L	10^{-6}	10^{-5}	10^{-4}	10^{-3}
$\partial^2_L(2E)$	$-0.5235988 \times 10^{18}$	$-0.5235988 \times 10^{15}$	$-0.5235988 \times 10^{12}$	-0.5235986×10^{9}
$2E$	-0.2617989×10^{6}	-0.2617944×10^{5}	-0.2617494×10^{4}	-0.2613008×10^{3}
L	0.01	0.1	1.0	10.0
$\partial^2_L(2E)$	-0.5235829×10^{6}	-0.5220101×10^{3}	-0.3897729	$-7.893493 \times 10^{-10}$
$2E$	-0.2569007×10^{2}	-2.182667	-4.666378×10^{-2}	0.0

The above data were obtained by using the following regularizing functions: $F(x) = \exp\{-x\}$, $F(x) = \exp\{-x^2\}$, $F(x) = \exp\{-x^3\}$, ..., $F(x) = \exp\{-x^6\}$, $F(x) = \exp\{-2\cosh(x) + 2\}$. For all these functions, the answers coincide to eight decimal places, which is excessive for our purposes.

As expected, the second derivative of the energy reaches the massless limit $\partial^2_L 2E_0 = -\pi/(6L^3) \cong -0.52359878/L^3$ as $L \to 0$ much faster than the energy reaches its massless limit $2E_0 = -\pi/(12L) \cong -0.26179939/L$.

THE THREE-DIMENSIONAL CASE

The method developed in the previous section is sure to operate in the case of a one-dimensional massive field. For it to be effective in the three-dimensional case, too, the number of differentiations must be doubled. Indeed, in the three-dimensional case the leading singular term is proportional to volume and thus has the form $c_{-4}L^3/\alpha^4$. To get rid of it, we must take the fourth derivative ∂^4_L.

The term that is logarithmic in α is sure to vanish after four differentiations. The transformation of the spectrum is constructed in such a way that the coefficient of the logarithmic divergence contains only positive powers of M. The negative power, M^{-1}, is multiplied by the sum $\sqrt{\omega_n^2 - \mu^2}$, which contains no logarithmic divergence according to the definition of μ in terms of (4).

For a massless scalar field in a sphere of unit radius $R = 1$ we get

$F(x)$	μ^2	μ	$2E_{\text{Cas}}/\mu$
$\exp\{-x^3\}$	0.01895799	0.1376880	0.0550299
$\exp\{-x^5\}$	0.01895777	0.1376872	0.0550527
$\exp\{-x^7\}$	0.01895777	0.1376872	0.0550531
$e^{-2\cosh x + 2}$	0.01895774	0.1376871	0.0550455

The values of the fourth derivative for different $F(x)$ coincide to four decimal places, with the largest deviation provided by $F(x) = \exp\{-x^3\}$. Accordingly, the accuracy in determining $E_{\text{Cas}} = 3.790 \times 10^{-5}$. The respective accuracy can also be estimated at four decimal places. This accuracy is not very high simply because the range of the used values of the regularizing parameter α corresponds to approximately 200 s-levels and, accordingly, to a maximal value of the orbital angular momentum l of approximately 600.

Note that in the process of our calculations we found not only the Casimir energy in the massless case $\mathcal{M} = 0$ but also the Casimir energies for all possible masses $\mathcal{M}$ in a range extending from zero to "effective" infinity:

$\mathcal{M}$	0	0.1	0.2	0.5
$2E(\mathcal{M})$	7.580×10^{-3}	7.372×10^{-3}	6.761×10^{-3}	4.236×10^{-3}
$\mathcal{M}$	1.0	2.0	5.0	10.0
$2E(\mathcal{M})$	1.649×10^{-3}	4.272×10^{-4}	9.843×10^{-5}	1.368×10^{-5}

The above results suggest that in such a renormalization procedure the vacuum energy has no IR singularity as $\mathcal{M} \to 0$,m in contrast to the results of [3, 14]. From the viewpoint of general theory, the appearance of IR singularities, as $\mathcal{M} \to 0$, in the vacuum energy for fields localized within a finite volume is highly problematic. Actually, the emergence of such singularities in [3, 14] stems from the fact that even when there is a "natural" renormalization point $\mathcal{M} \to 0$, there still remains a certain arbitrariness in the computational procedure. Precisely, the final expression for the energy may acquire a function that rapidly decreases as $\mathcal{M} \to \infty$ but has a logarithmic singularity as $\mathcal{M} \to 0$. In contrast, our renormalization procedure does not involve subtractions and, hence, contains no ambiguities: in the differentiation process only positive powers of M vanish, since there is no way in which they may appear in the final expression for the Casimir energy. Here, in the entire range of M (from μ to ∞), the derivative is finite and decreases, as $M \to \infty$, faster than M^{-5} (although slower than the exponential function; this was first noted in [14]), so that at $M = \mu$, i.e., at $\mathcal{M} = 0$ we have the final result.

CONCLUSION

Thus, we have suggested a numerical method for finding the Casimir energy that has no subtraction and, hence, does not require normalization to remove indeterminate forms in the computational procedure. Note, however, that this is only one possible approach to renormalization.

One drawback of the method is the need to do calculations with an elevated number of true decimal places (in particular, the system's spectrum must be found to a high accuracy). At the same time, a clear merit of the method is its "ideological" and technical simplicity, which makes it possible to use it when there is no analytical expression for the spectrum (and no analytical expression for an equation that could be used to find the levels in the spectrum). More than that, there is no need to calculate approximate asymptotic expressions for the levels.

Our method can be used when there is no explicit transcendental equation for the levels. Moreover, in the case of three-dimensional problems it is advisable to use this method for relatively simple spectra, too. The thing is that in three-dimensional problems (even in spherically symmetric problems) one has to sum at least a double series, and even for trivial systems renormalization is extremely cumbersome [14].

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