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General Properties of Geophysical Hydrodynamics Models

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Abstract—The influence of simplifying assumptions used in geophysical hydrodynamics models on the general properties of space and time, as well as on the fundamental physical principles that underlie the mechanical systems, is studied. Analysis of the general properties of such models is supplemented by investigation of the peculiarities introduced by stratification into the solutions of nonlinear stationary models. The possibilities for representing the solutions as finite and infinite Fourier series in cylindrical coordinates are explored.

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Since many of the scalar parameters responsible for specific aspects of geophysical flows are small, the physical processes in most studies are analyzed using only a few dynamic factors (models of an ideal fluid are considered, buoyancy and/or rotation effects are taken into account, or the properties of geophysical flows are analyzed using steady-state waves and stationary models). However, it has been shown in several cases that, although the scalar parameters are small, the properties of flows described by similar models can differ significantly or coincide, depending on the chosen range of the space–time domain. Thus, studying the structural stability of both general properties of physical models and individual flows or some of their elements with respect to the choice of decisive factors of the model turns out to be a topical problem. In this paper, we study the role of simplifying assumptions used in geophysical hydrodynamics models and their influence on the fundamental physical principles that underlie the mechanical systems and the peculiarities of the solutions.

Symmetries of the general equations for geophysical flows. For an incompressible stratified fluid in the Boussinesq approximation, the equations of motion, including the Earth's rotation with an angular velocity Ω, viscosity, and diffusion are [1–3]

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} &= -\nabla P + \nu\Delta\mathbf{u} + S\mathbf{g} - 2\Omega\times\mathbf{u}, \\ S_t + (\mathbf{u}\nabla)S &= \kappa\Delta S, \quad \operatorname{div}\mathbf{u} = 0. \end{aligned} \tag{1}$$

Here, $\mathbf{u} = (u, v, w)$ is the velocity, S is the salinity, P is the pressure per unit density, $\Omega = \Omega\mathbf{e}_z$ is the Earth's angular velocity, $\Omega\times\mathbf{u} = \Omega(-\mathbf{e}_x v + \mathbf{e}_y u)$, $\mathbf{g}$ is the gravity, and ν and κ are the coefficients of kinematic viscosity and salt diffusion, respectively.

Symmetries of the spherical and planar Earth models. Two basic types of models are used in geophysical hydrodynamics: global ones, with flows in a spherical geometry in a centrally symmetric field of gravity $\mathbf{g} = -g\mathbf{e}_r$, and local ones in the approximation of a planar Earth and in a uniform field of gravity $\mathbf{g} = -g\mathbf{e}_z$. However, apart from the natural assumptions that a model must be chosen by taking into account the flow scale, the properties of the models have not yet been compared in detail. The second commonly used approximation of the hydrodynamics of inhomogeneous media is the Boussinesq approximation. The magnitude of the buoyancy effects is generally characterized by the stratification scale, $\Lambda = (d\ln\rho/dz)^{-1}$, and the buoyancy frequency, $N = \sqrt{g/\Lambda}$. In writing system (1), we took into account the fact that the stratification is generally weak and the change in density due to the salinity may be neglected everywhere except the term containing the gravity. When passing to the original system of equations of motion for an incompressible fluid, we must rewrite the first equation of system (1) as [1, 3]

$$\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} = -\frac{1}{\rho(S)}\nabla P + \nu\Delta\mathbf{u} + \mathbf{g} - 2\Omega\times\mathbf{u}. \tag{2}$$

Continuous groups are a natural means of analyzing the general properties of physical models. In a spherically symmetric field of gravity $\mathbf{g} = -g\mathbf{e}_r$ ($x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$) with the Oz rotation axis, the symmetry group is generated by the generators that reflect such fundamental properties of the physical systems as the time homogeneity $X_1 = \partial_t$ (time shifts), the relationship between pressure and salinity $X_2 = \partial_S - gr\partial_P$, and the freedom in choosing the pressure for an arbitrary function of time $X_\pi = \pi(t)\partial_P$. These properties are common to all of the above models. In addition, irrespective of whether the Boussinesq approximation is used, the global geophysical models have a symmetry of rotation relative to the vertical $Y_1 = \partial_\varphi$ axis (rotations about the Oz axis).

The remaining properties of the spherical and planar models differ significantly. Since the field of gravity is centrally symmetric, the global models in the Boussinesq approximation have spatial isotropy relative to the O$\tilde{x}$ and O$\tilde{y}$ axes rotating with the Earth with an angular velocity Ω in a noninertial coordinate system. The generators of the transformation groups are

the $O\tilde{y}$ axis:

$$\begin{aligned} Y_2 &= -2\cos\tilde{\varphi}\partial_\vartheta + 2\cot\vartheta\sin\tilde{\varphi}\partial_\varphi \\ &+ 2\sin\tilde{\varphi}\left(r\Omega + \frac{1}{\sin\vartheta}w\right)\partial_u \\ &+ 2\left(r\Omega\cos\vartheta\cos\tilde{\varphi} - \frac{1}{\sin\vartheta}u\sin\tilde{\varphi}\right)\partial_w \\ &+ (r\Omega)^2\sin 2\vartheta\cos\tilde{\varphi}\partial_P, \end{aligned} \tag{3}$$

the $O\tilde{x}$ axis:

$$\begin{aligned} Y_3 &= 2\sin\tilde{\varphi}\partial_\vartheta + 2\cot\vartheta\cos\tilde{\varphi}\partial_\varphi \\ &+ \cos\tilde{\varphi}\left(r\Omega + \frac{2}{\sin\vartheta}w\right)\partial_u \\ &- \left(r\Omega\cos\vartheta\sin\tilde{\varphi} + \frac{2}{\sin\vartheta}u\cos\tilde{\varphi}\right)\partial_w \\ &- (r\Omega)^2\sin 2\vartheta\sin\tilde{\varphi}\partial_P, \end{aligned}$$

where $\tilde{\varphi} = \varphi + \Omega t$, $r\sin\vartheta\sin\tilde{\varphi} = \tilde{y} = \text{inv}$, and $r\sin\vartheta\cos\tilde{\varphi} = \tilde{x} = \text{inv}$.

Since the two uniform fields of Coriolis forces and gravity act in different directions, the planar Earth models do not admit the spatial isotropy properties at all, which is physically unjustified. Using a constant (in direction) field of gravity increases the number of symmetries of the local models through another fundamental principle—the Galilean relativity principle extended to include all translationally moving coordinate systems:

$$\begin{aligned} X_\chi &= \chi(t)\partial_x + \dot{\chi}(t)\partial_u - (\ddot{\chi}(r)x + \dot{\chi}(t)\Omega y)\partial_P, \\ X_\eta &= \eta(t)\partial_y + \dot{\eta}(t)\partial_v - (\ddot{\eta}(t)y - \dot{\eta}(t)\Omega x)\partial_P, \\ X_\zeta &= \zeta(t)\partial_z + \dot{\zeta}(t)\partial_w - \ddot{\zeta}(t)z\partial_P, \end{aligned} \tag{4}$$

where $\chi(t)$, $\eta(t)$, and $\zeta(t)$ are arbitrary functions of time; the law of motion of the coordinate system relative to the original one in the x, y, and z directions, respectively, is the physical content of the latter.

At constant values of the functions χ, η, and ζ, generators (4) transform into the generators of the spatial translation groups. The standard Galilean transformations follow from (4) for a linear time dependence of the arbitrary functions. Although the Coriolis forces are nonpotential in nature, the forces of inertia that arise when passing to a noninertial coordinate system do not change the form of the equations of motion, but lead only to a pressure renormalization. This effect is similar to the transition from the total pressure and salinity to dynamical variables through the cancellation of the hydrostatic components.

The models with a spherically symmetric field of gravity imply fixing the position of the origin, which leads to the loss of the symmetries related to spatial homogeneity. An additional symmetry in this case is the rotation-modified symmetry of the self-similar transformations with the generator

$$\begin{aligned} Z_1 &= 2t\partial_t + r\partial_r - 2\Omega t\partial_\varphi - v\partial_v - u\partial_u \\ &- (2\Omega r\sin\vartheta + w)\partial_w - 3S\partial_S - 2(P + r^2\Omega^2\sin^2\vartheta)\partial_P. \end{aligned} \tag{5}$$

Many of the hydrodynamic models have the symmetries of self-similar transformations and they are widely used to construct exact solutions. In the case of planar geometry, rotation makes the presence of any extension symmetries in the model impossible.

Complete equations of motion and the Boussinesq approximation. The peculiarities introduced by the Boussinesq approximation into geophysical models were studied using the equations for a stratified fluid written in the absence of Coriolis forces for a uniform field of gravity as an example (Eqs. (1) and (2) at $\Omega = 0$). Since the full list of symmetries of the equations for stratified flows is given in [4], here we note only the most significant changes introduced by the Boussinesq approximation.

A distinctive feature of the general equations for an incompressible stratified fluid (2) is that they closely follow the Galilean relativity principle without any possibility of its broad interpretation (4). In this case, the generators of the transformation groups coincide with those of the hydrodynamic equations and are

the Galilean transformations:

$$X_{3,4,5} = t\partial_{x_i} + \partial_{u_i}, \quad i = 1, 2, 3. \tag{6}$$

Analysis shows that the extension of symmetry (6) for Eqs. (1) takes place because the dependence of the density on salinity in the pressure-containing term is neglected and because the property of barotropy, which by no means all flows possess, is imparted to a stratified fluid.

Another distinctive feature of the Boussinesq approximation is the establishment of a difference between the gravitational and inertial masses. As a result, anisotropy appears when we pass to a freely falling coordinate system. Whereas the original model is invariant with respect to three rotation groups in the horizontal and vertical (with a modification) planes,

rotations in the horizontal,

$$Y_4 = y\partial_x - x\partial_y + v\partial_u - u\partial_v, \tag{7}$$

and vertical,

$$Y_{5,6} = \left(\frac{1}{2}gt^2 + z\right)\partial_{x_i} - x_i\partial_z + (gt + w)\partial_{u_i} - u_i\partial_w,$$

planes, Eqs. (1) are invariant only with respect to rotations in the horizontal plane.

Thus, both the approximation of a uniform field of gravity and the Boussinesq approximation introduce significant changes into the basic properties of geophysical flow models. Although this fact per se does not provide a basis for abandoning the use of simplified models, increased demands should be imposed on the extrapolation of the results obtained on their basis to natural systems. The analysis in the next section is another example of a singular dependence of the properties of the solutions on weak effects that do not affect the order of the equations of motion.

The influence of stratification on the separation of variables and the properties of the solutions to the two-dimensional steady-state equations for an incompressible fluid. Although the construction of steady-state solutions for flows with nonzero vorticity in a homogeneous fluid is a classical hydrodynamic problem [5], some aspects of the self-similar solutions in the theory of vortex flows are still the subject of study [6]. In this paper, using the steady-state two-dimensional equations for an ideal incompressible fluid as an example, we consider the construction of solutions for vortex flows by the separation of variables in cylindrical (r, φ) coordinates and study the uniformity of the passage to the limit from a stratified medium to a homogeneous one.

Using the invariance of the equations for an ideal homogeneous fluid with respect to rotations, we will also seek steady-state solutions in the form of expansions into a Fourier series in the angular variable

$$\psi = \sum_{n=0} \psi_n \exp(in\varphi). \tag{8}$$

To interpret the results, we use two forms of the equations of motion written in stream function–salinity variables. The salt transport equation

$$\{\psi_\varphi \partial_r - \psi_r \partial_\varphi\} S = 0 \tag{9}$$

will be immediately assumed to have been integrated based on the widely used assumption that the stream function is linearly related to the stratifying component $S = S_0\psi$. The equation for vorticity $\omega = -\Delta\psi$ will be singly integrated in one case and will remain unchanged in the other case. Given the last remarks, let us write the two forms of the equations of motion

$$\{\psi_\varphi \partial_r - \psi_r \partial_\varphi\}\Delta\psi + \frac{S_0}{\mathrm{Fr}}[r\cos\varphi\partial_r - \sin\varphi\partial_\varphi]\psi = 0, \tag{10}$$

$$\Delta\psi = F(\psi) + \frac{S_0}{\mathrm{Fr}} r\sin\varphi, \tag{11}$$

where F is an arbitrary function of ψ, Fr is the Froude number, and $\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\varphi^2$ is the Laplacian in cylindrical coordinates.

During the solution for a homogeneous fluid, the vorticity will be assumed to be nonzero. For a stratified fluid, this automatically holds due to the presence of an additional term in Eq. (11). The constant of integration S_0 enters into Eqs. (10) and (11) in a regular way so that the equations for a homogeneous fluid without dissipation are derived from the equations for a stratified fluid by setting S_0 equal to zero.

Solutions of the equations for a homogeneous fluid. Monomial solutions. Equation (10) for a homogeneous fluid does not contain the angular variable explicitly and, hence, is invariant with respect to shifts in φ. We can then seek a solution in the form of separable variables

$$\psi = \psi_n(r)\exp(in\varphi). \tag{12}$$

Substituting (12) into (10) for the radial part of the solution yields the Bessel equation

$$\frac{(L_n\psi_n)_r}{L_n\psi_n} = \frac{\psi_{n,r}}{\psi_n} \text{ or } L_n\psi_n = \pm\lambda^2\psi_n, \tag{13}$$

where $L_n = \partial_r^2 + r^{-1}\partial_r - (n/r)^2$ and λ is an arbitrary constant.

Depending on the values of λ, the solutions to Eq. (13) will be integral-order Bessel functions of real, $\{J_n(\lambda r), Y_n(\lambda r)\}$, imaginary, $\{K_n(\lambda r), I_n(\lambda r)\}$, and, in general, complex, $\{H_n^{(1,2)}(\lambda r)\}$, arguments. In contrast to the linear equations, the argument of the eigenfunctions for the solutions of nonlinear equation (10) depends on the constant of integration λ. In view of nonlinearity, the solution superposition principle loses its generality and conditions for the compatibility of Eq. (10) with linear combinations of solutions to (13) taken for different values of the parameter n must be determined.

Binomial expansions. Let the solution to Eq. (10) be a linear combination of the functions

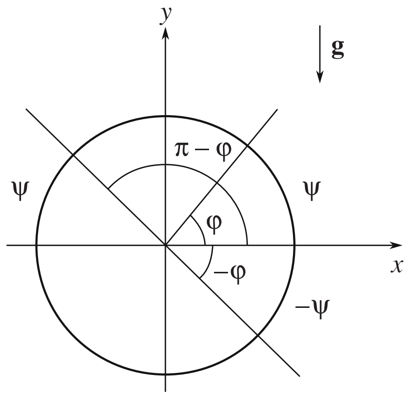
$$\psi = \psi_n(r)\exp(in\varphi) + \psi_m(r)\exp(im\varphi). \tag{14}$$

As above, the variable separation condition restricts the radial components of solutions (14) to Bessel functions,

$$L_n\psi_n = \pm\lambda^2\psi_n \text{ and } L_m\psi_m = \pm\mu^2\psi_m, \tag{15}$$

and the solutions with $\mu = \lambda$ will satisfy the superposition principle.

The solutions consisting of any finite number of terms can be constructed in a similar way. Using condition (13), we can show that the solutions consisting of a finite number of modes correspond to a linear relation between vorticity and the stream function ($F(\psi) = \pm\lambda^2\psi$ in Eq. (11)). An infinite number of terms in the Fourier series (8) corresponds to nonlinear relations (the properly nonlinear peculiarities of the hydrodynamic equations). After the substitution of (8) into Eq. (10), the coefficients of the series are determined recursively; the



Discrete symmetries of the steady-state equations for a stratified fluid.

recurrence equations for the zero term are satisfied identically (a corollary of the incompressibility condition for the two-dimensional problem):

$$n[\psi_n(L_0\psi_0)_r - \psi_{0,r}L_n\psi_n] = \sum_{m=1}^{n-1}[(n-m)\psi_{m,r}L_{n-m}\psi_{n-m} - m\psi_m(L_{n-m}\psi_{n-m})_r], \quad n = 1, 2, 3, \ldots, \tag{16}$$

where $\psi_0(r)$ is an arbitrary function of r.

The recurrence (second-order) equations (16) show that solution (8) was determined for one arbitrary function; the remaining terms of the series depend significantly on the choice of ψ_0. Therefore, solution (8) may be said to be represented by the function ψ_0. If ψ_0 is a Bessel function, then a linear relation between vorticity and stream function is again valid.

Solutions of the equations for a stratified fluid. Let us consider the peculiarities introduced into the vortex flows of an ideal fluid by the buoyancy forces in the case where the salinity is related to the stream function by a linear relation. In view of the preferred orientation of the vertical axis, a single-mode representation of the solution in the form

$$\psi = \psi_1(r)\sin\varphi$$

is possible. Substituting the latter into Eq. (10) yields

$$\{\psi_1\partial_r - \psi_{1,r}\}L_1\psi_1 + \frac{S_0}{\mathrm{Fr}}(r\partial_r - 1)\psi_1 = 0. \tag{17}$$

Comparison of Eq. (17) with Eq. (15) for a homogeneous fluid shows that the solution of (17) is not restricted only to Bessel functions. However, since the linear function of r is of particular importance for the operator of Eq. (17), a solution can again be rapidly constructed using the representation

$$\psi_1 = Ar + f_1, \tag{18}$$

where the function $f_1(r)$ satisfies Eq. (15) at $n = 1$ with the constant of integration λ and A is a constant to be determined. Substituting (18) into Eq. (17) using the relation $L_1\psi_1 = \pm\lambda^2 f_1$ yields

$$A = \mp S_0/\mathrm{Fr}\lambda^2.$$

Solution (18) corresponds to a linear relation between vorticity, stream function, and salinity. The separated linear part in the stream function has the clear physical meaning of original stratification; f_1 then has the meaning of a perturbation, which is assumed to be small in the Boussinesq approximation. Polynomial solutions can be constructed based on the singly integrated equation of motion (11), where the following linear function is chosen as F:

$$\Delta\psi = \pm\lambda^2\psi + \frac{S_0}{\mathrm{Fr}}r\sin\varphi.$$

As in the case of a homogeneous fluid, the nonlinear properties of the equations for a stratified fluid can be studied only based on an infinite number of harmonics, i.e., using the entire Fourier series. Before constructing a solution, let us study the symmetry properties of Eq. (10). The substitution $\varphi \longrightarrow -\varphi$ and the requirement that the defining equation be invariable express the physically justified "top–bottom" symmetry of a stratified fluid (to sink is as difficult as to rise) as the oddness condition for the stream function, $\psi(-\varphi) = -\psi(\varphi)$. Another symmetry, the symmetry of reflection relative to the Oy axis, does not change the vertical positions of the fluid particles. As a result, the stream function is "even" with respect to the substitution $\varphi \longrightarrow \pi - \varphi$, $\psi(\pi - \varphi) = \psi(\varphi)$ (see the figure). Using symmetry properties, we can find, without solving the problem, that the expansion into a Fourier series must be performed only in sines (the symmetry relative to the Ox axis) and, moreover, only the odd terms of the series will be nonzero (the symmetry relative to the Oy axis):

$$\psi = \sum_{n=0}^{\infty}\psi_n\sin((2n+1)\varphi). \tag{19}$$

Thus, using steady-state two-dimensional stratified flows as an example, we showed that nonlinear peculiarities of the equations give rise to an infinite number of harmonics in the solution. In the widely used case of a linear relation between vorticity and stream function, the solutions to the equations for a stratified fluid are again described by Bessel functions, as in the case of a homogeneous fluid.

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