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A Physically Realizable Whitening Filter for Cryogenic Resonant Gravitational-Wave Antennas

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Abstract—In this paper the problem of a physically realizable whitening-filter (WF) synthesis for cryogenic resonant gravitational-wave antennas is considered in the framework of the optimal linear Kalman–Bucy filtering. A system of equations determining the structure of the Kalman filter for gravitational-wave antennas with a displacement converter is presented. The transfer function of a stationary physically realizable WF is derived. The Kalman approach ensures the possibility of high frequency parameter measurements of Gaussian and non-Gausssian noises in low-dissipation oscillatory systems for observation intervals, which are relatively short in comparison to the relaxation time.

Key words: Resonant gravitational-wave antennas, Kalman–Bucy optimal filtering, whitening filter.

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1. INTRODUCTION

Cryogenic resonant gravitational-wave antennas (RGA) [1] containing a mechanical displacement converter belong to the class of linear stationary systems with two degrees of freedom. In such devices the spectrum of the random process $x(t)$ in the line-channel output is confined (for positive frequencies) to a narrow band (ω_{min}, ω_{max}). Therefore, using the complex representation of arbitrary quasi-harmonic signals, we have

$$x(t) = \lambda s(t) + n(t) = \mathrm{Re}[\tilde{x}(t)\exp\{j\omega_0 t\}],$$

where $\lambda = \{0, 1\}$ is the detection parameter, $s(t)$ is the desired signal, $n(t)$ is an additive noise, $\omega_0 = (\omega_{max} + \omega_{min})/2$ is the central frequency, $\omega_0 \approx 2\pi \times 10^3$ rad/sec, and $\tilde{x}(t)$ is the complex envelope of the random process $x(t)$, whose spectrum is confined to a narrow band ($-\Delta_\omega$, Δ_ω), $\Delta_\omega = (\omega_{max} - \omega_{min})/2$.

Preliminary information processing in the "fast filtering" regime is performed according to the energy criterion of the signal-to-noise ratio via the scheme WF–MF, where WF and MF are the whitening and the matched filters, respectively. The transfer function of the optimal WF–MF system is given by the following expression [2]

$$H_{\mathrm{WF-MF}}(j\omega) = c\frac{G^*(j\omega)}{N(\omega)}\exp\{-j\omega t_d\} = H_{\mathrm{WF}}(\omega)H_{\mathrm{MF}}(j\omega),$$

where c is a constant, $G(j\omega)$ is the line-channel transfer function, $N(\omega)$ is the spectral density of the additive noise $n(t)$, t_d is the time delay,

$$H_{\mathrm{WF}}(\omega) = \frac{1}{\sqrt{N(\omega)}},\quad H_{\mathrm{MF}}(j\omega) = c\frac{G^*(j\omega)}{\sqrt{N(\omega)}}\exp\{-j\omega t_d\} \tag{1}$$

are the WF and MF transfer functions.

A linear system with the transfer function (1) is not a physically realizable one: the random process in the output of the MF at time t is determined by the external impact $x(t)$ where $t \in (-\infty, \infty)$.

In practice, see [1], while designing optimal adaptive filters the spectral density $N(\omega)$ is replaced by a non-parametric estimate $\hat{N}(\omega)$. For the correlation time $\tau^* \approx 1400$–2000 sec and the stationarity period $T_s \simeq 2$ h, such an approach does not seem to be appropriate. Indeed, for a sufficiently small spectral gap $W(\omega; \vartheta)$ of the width $\sim 2\pi/\vartheta$, the bias $\Delta(\omega)$ and the variance $D(\omega)$ of the estimate $\hat{N}(\omega)$ are given by the following expression, see [3]:

$$\Delta(\omega) \simeq K_\Delta N''(\omega),\quad D(\omega) \simeq (K_D/T)N^2(\omega),$$

where T is the length of the observation interval (t_0, $t_0 + T$),

$$K_\Delta = \frac{1}{4\pi}\int_{-\infty}^{\infty}\omega^2 W(\omega; \vartheta)d\omega,$$

$$K_D(\vartheta) = (1/2\pi)\int_{-\infty}^{\infty}W^2(\omega; \vartheta)d\omega.$$

The parameter ϑ is chosen taking into account contradictory requirements. For "large" ϑ (the narrow spectral gap) the estimate bias $\Delta(\omega)$ decreases. On the other hand, in order to reduce the variance $D(\omega)$ of the estimator we have to use a wide spectral gap with "small" ϑ. In practice, taking into account that $D(\omega) \propto T^{-1}$, we achieve a sensible compromise by increasing the length T of the observation interval. Significant distortions during the spectral density $N(\omega)$ measurements are unavoidable if $T \simeq T_s \approx (3.6–5.1)\tau^*$.

The impossibility of the optimal choice for the value of the spectral window $W(\cdot)$ within the "Italian" scheme of information processing stimulates the development of alternative algorithms for a preliminary statistical analysis of the output signals of the cryogenic resonant gravitational-wave antennas. In the present paper we discuss the particular features of designing (synthesizing) the main element of optimal filters—a physically realizable WF—for short observation intervals with $T \ll \tau^*$.

2. WF SYNTHESIS (THE GENERAL CASE)

A resonant gravitational-wave antenna containing a mechanical displacement converter can be considered as a linear system formed by two connected mechanical pendulums. Let M_i, K_i and H_i be the equivalent mass, rigidity, and the friction coefficient of the pendulums, $i = 1, 2$ ($i = 1$ corresponds to the "gravitational detector", $i = 2$ corresponds to the "displacement converter"). Then in the "no signal" state ($\lambda = 0$) we have [1]

$$\begin{cases} (p^2 + \Omega_1^2)x_{10}(t) - \varepsilon\omega_2^2 x_{20}(t) = f_1(t), \\ -\omega_2^2 x_{10}(t) + (p^2 + \omega_2^2)x_{20}(t) = f_2(t), \quad 0 \le t \le T, \\ x_0(t) = x_{20}(t) + x_a(t), \end{cases} \quad (2)$$

where $x_{1\lambda}(t)$ and $x_{2\lambda}(t)$ are the generalized coordinates in the state λ, $x_\lambda(t)$ is the resulting random process at the registration system input, $p = d/dt$ is the differentiation operator, $\Omega_1^2 = \omega_1^2 + \varepsilon\omega_2^2$, $\omega_i^2 = K_i/M_i$, $i = 1, 2$, $\varepsilon = M_2/M_1 \ll 1$, $T \ll 2\min[M_1/H_1, M_2/H_2]$.

According to the fluctuation-dissipation theorem in the Langevin description of the thermal noises, see [4], the correlation functions of the external sources $f_1(t)$ and $f_2(t)$ are defined by the following relationship:

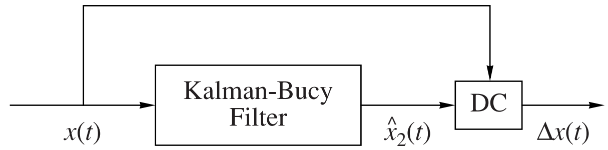
$$\langle f_i(t) f_k(\tau)\rangle = 2\kappa T_0 B_{ik}\delta(t-\tau), \quad (3)$$

where $\langle\cdot\rangle$ denotes the statistical averaging, κ is the Boltzmann constant, T_0 is the thermostat temperature,

$$B_{11} = (H_1 + H_2)/M_1^2,$$

$$B_{12} = -H_2/(M_1 M_2), \text{ and } B_{22} = H_2/M_2^2.$$

In the analysis presented below we will model the additive noise $x_a(t)$ by a wideband random process of



The equivalent scheme of a non-stationary WF within the Kalman framework.

the Gaussian white-noise type with the correlation function

$$\langle x_a(t) x_a(\tau)\rangle = N_a\delta(t-\tau). \quad (4)$$

One can show, see [2, 5], that, in the general case, a physically feasible WF can be realized in the form of a linear non-stationary system with a feedback. The block-scheme of such a system is shown in the Figure, where $\hat{x}_{20}(t)$ is the optimal (in the sense of the minimal error variance) estimate of the random process $x_{20}(t)$ realization on the background of the delta-correlated additive noise $x_a(t)$, DC is the differential cascade, $x_\Delta(t) = x_0(t) - \hat{x}_{20}(t)$ is the Gaussian white noise with the correlation function (4). Estimation of the correlated (colored) noise is carried out with a help of a Kalman filter.

Within the Kalman–Bucy optimal filtering theory, see [5], the equations of motion (2) are written down in the vector form:

$$x_0(t) = \mathbf{C}y_0(t) + x_a(t),$$

where $\mathbf{C} = (1\ 0\ 0\ 0)$, and $y_0(t) = [y_{10}(t)y_{20}(t)y_{30}(t)y_{40}(t)]^{\mathrm{T}}$ is the state vector. The components of the vector $y_0(t)$ are connected with the generalized coordinates $x_{10}(t)$ and $x_{20}(t)$ of the mechanical system by the following relationships: $y_{10}(t) = x_{10}(t)$, $y_{20}(t) = px_{10}(t)$, $y_{30}(t) = x_{20}(t)$, and $y_{40}(t) = px_{20}(t)$. Using Eq. (2) and (3) we obtain

$$p\mathbf{y}_0(t) = \mathbf{P}y_0(t) + f(t), \quad \langle f(t) f^{\mathrm{T}}(\tau)\rangle = \mathbf{Q}\delta(t-\tau).$$

Here

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\Omega_1^2 & 0 & \varepsilon\omega_2^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_2^2 & 0 & -\omega_2^2 & 0 \end{bmatrix}, \quad \mathbf{Q} = 2\kappa T\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B_{11} & 0 & B_{12} \\ 0 & 0 & 0 & 0 \\ 0 & B_{12} & 0 & B_{22} \end{bmatrix}, \quad (5)$$

and $f(t) = [0\, f_1(t)\, 0\, f_2(t)]^{\top}$ is a vector random process.

The Kalman filter is determined by the following system of matrix equations, see [5],

$$\left.\begin{aligned} p\hat{y}_0(t) &= P\hat{y}_0(t) + z(t)[x(t) - C\hat{y}_0(t)], \\ z(t) &= e(t)C^{\top}N_a^{-1}, \\ pe(t) &= Pe(t) + e(t)P^{\top} - e(t)C^{\top}N_a^{-1}Ce(t) + Q, \end{aligned}\right\} \quad (6)$$

where $\hat{y}_0(t)$ is the optimal estimate of the vector random process $y_0(t)$, $z(t)$ is the transfer ratio, $e(t) = [e_{ik}(t)]$ is the dispersion matrix, $e_{ik}(t) = \langle[y_{i0}(t) - \hat{y}_{k0}(t)]^2\rangle$.

The matrix dispersion equation in (6) is equivalent to $(l/2)(l + 1)$ scalar nonlinear differential equations, where $l = 4$ is the dimensionality of the state vector $y_0(t)$ (for the choice of the initial condition see [5]). The dispersion equation in the cases $l \geq 2$ is solved using numerical methods.

3. STATIONARY PHYSICALLY REALIZABLE WF (A SPECTRAL APPROACH)

In the general case the WH, which block-scheme is shown on the Figure, is a linear system with variable coefficients. However, since the matrices **P**, **Q**, (5) and **C** do not depend on t, the parameters of the Kalman filter converge to constant values [5] as t increases. In this situation the Kalman filter approaches a physically realizable stationary Wiener filter, see [5]. The last filter contains a physically realizable WF with the transfer function $H_{\mathrm{WF}}^{(\mathrm{p})}(j\omega)$, whose modulus is given by the following expression

$$\left|H_{\mathrm{WF}}^{(\mathrm{p})}(j\omega)\right| = H_{\mathrm{WF}}(\omega) = \frac{1}{\sqrt{N(\omega)}}.$$

The computation of the transfer function for a physically realizable WF from its modulus is explained in detail in the monograph [5].

In the framework of the Langevin approach the spectral density $N(\omega)$ of the random process $x_0(t)$ is determined by the following expression, see [6],

$$N(\omega) = 2\kappa T H_1 |G(j\omega)|^2 \Gamma(\omega) \propto |G(j\omega)|^2 \Gamma(\omega),$$

where $\Gamma(\omega)$ is the differential noise factor, which in the case of the gravitational-wave antenna with a displacement converter is a polynomial of the following type

$$\Gamma(\omega) = a_0 + a_2\omega^2 + \ldots + a_8\omega^8, \quad |\omega| \in (\omega_{\min}, \omega_{\max}). \tag{7}$$

Let ω_k^+, $k = \overline{1, 4}$, be the solution of the algebraic equation $\Gamma(\omega) = 0$ with non-negative imaginary parts. Then

$$\Gamma(\omega) = \left|\Psi^+(j\omega)\right|^2, \quad \Psi^+(j\omega) = \prod_{k=1}^{4}(\omega - \omega_k^+), \quad N(\omega) \propto \left|G(j\omega)\Psi^+(j\omega)\right|^2. \tag{8}$$

The multiplier $G(j\omega)$ in Eq. (8) is the transfer function of the RGA line channel, the second multiplier $\Psi^+(j\omega)$ can be interpreted as the transfer function of some physically realizable linear system, see [5]. Therefore,

$$H_{\mathrm{WF}}^{(\mathrm{p})}(j\omega) \propto [G(j\omega)\Psi^+(j\omega)]^{-1}.$$

The WF contains an inertialess inverse filter (IF) with the transfer function

$$G^{-1}(p) = (p^2 + \omega_{\mathrm{I}}^2)(p^2 + \omega_{\mathrm{II}}^2)/\omega_2^2, \tag{9}$$

where (see above) ω_{I} and ω_{II} are the eigenfrequencies of the mechanical system,

$$\omega_{\mathrm{I,\,II}} = \sqrt{\frac{(\Omega_1^2 + \omega_2^2) \pm D}{2}}, \quad D = \sqrt{(\Omega_1^2 - \omega_2^2)^2 + \varepsilon\omega_2^4}.$$

In practice the eigenfrequencies ω_{I} and ω_{II} are found with high precision during the calibration process.

The bandwidth of the linear channels in cryogenic RGA is confined to a narrow interval around the central frequency $\omega_0 = (\omega_{\mathrm{I}} + \omega_{\mathrm{II}})/2$ (see above). For the purpose of a general analysis it is appropriate to replace such systems by a low-frequency equivalent. The noise factor $\Gamma_0(\omega)$ of such a low-frequency equivalent is connected with the noise factor $\Gamma(\omega)$ of the original system as follows: $\Gamma_0(\omega) = G(\omega_0 + \omega)$, $|\omega| \ll \omega_0$. Taking into account Eq. (8) we have

$$\Gamma_0(\omega) = \left|\Psi_0^+(j\omega)\right|^2, \quad \Psi_0^+(j\omega) = \Psi^+[j(\omega_0 + \omega)], \quad |\omega| \ll \omega_0. \tag{10}$$

If the partial frequencies are identical $\omega_1 = \omega_2 = \omega_0$, then the noise factor $\Gamma_0(\omega)$ can be written down as follows, see [6],

$$\Gamma_0(\omega) \approx 1 + \frac{(\omega^2 - \Delta_b^2)^2}{(\omega_0\Omega_{\min})^2} + \frac{Q_1}{Q_2}\left(\frac{\omega}{\Delta_b}\right)^2,$$

where $\Delta_b = (\omega_2 - \omega_1)/2$ is the beat frequency ($|\Delta_b| \ll \omega_0$), Q_1 and Q_2 are the Q-factors of the gravitational-wave detector and the displacement converter, respectively ($Q_2 \lesssim Q_1$), $\Omega_{\min}$ is the line-channel bandwidth of the RGA without a displacement converter. Therefore, taking into account Eq. (10) we obtain

$$\Psi_0^+(j\omega) \propto (\omega - \omega_{1;\,0}^+)(\omega - \omega_{2;\,0}^+),$$

where $\omega_{i,\,0}^+$, $i = 1, 2$, are the roots of the biquadratic algebraic equation $\Gamma_0(\omega) = 0$, with non-negative imaginary parts. The solutions of the equation $\Gamma_0(\omega) = 0$ depend on the system parameters. After some transformations we obtain:

Case A: $\xi^2 < (2\nu + 1)/\nu^2 \longrightarrow \omega_{1,\,2;\,0}^+ = \pm\Delta_{\mathrm{A}} + j\gamma_{\mathrm{A}}$,

Case B: $\xi^2 > (2\nu + 1)/\nu^2 \longrightarrow \omega_{1,\,2;\,0}^+ = j\gamma_{\mathrm{B},\,1,\,2}$.

Here

$$\Delta_{\mathrm{A}} = \Delta_b\sqrt{\frac{1}{2}[\vartheta(\xi) + 1 - \nu\xi^2]},$$

$$\gamma_A = \Delta_b \sqrt{\frac{1}{2}[\vartheta(\xi) - 1 + \nu\xi^2]}, \quad (11)$$

$$\gamma_{B,1,2} = \sqrt{(1 - \nu\xi^2) \pm \xi(\nu^2\xi^2 - 2\nu - 1)^{1/2}},$$

where $\xi = (\omega_0\Omega_{min}/\Delta_b^2)$, $2\nu = Q_1/Q_2 \gtrsim 1$, $\vartheta(\xi) = \sqrt{(1+\xi^2)}$.

Using Eqs. (10) and (11) we obtain

$$\frac{1}{[\Psi_0^+(j\omega)]_A} \propto \frac{1}{(\omega - \Delta_A + j\gamma_A)(\omega + \Delta_A + j\gamma_A)},$$
$$\frac{1}{[\Psi_0^+(j\omega)]_B} \propto \frac{1}{(\omega - j\gamma_{B,1})(\omega - j\gamma_{B,2})}. \quad (12)$$

Let $H(j\omega)$ be the transfer function of a consecutive high-Q oscillatory circuit with the resonance frequency ω_r and the logarithmic decrement $\gamma_r \ll \omega_r$. Then

$$H_0(j\omega) = H[j(\omega + \omega_0)] \approx -\left(\frac{\omega_0}{2}\right)\frac{1}{(\Delta_r + \omega - j\gamma_r)}, \quad (13)$$

where $|\omega| \ll \omega_0 = \Delta_r + \omega_r$, $\Delta_r \ll \omega_r$.

Taking into account Eqs. (11), (12), and (13) we obtain the block-scheme of a stationary physically realizable WF in the "fast filtering" regime:

$$\text{IF} - \text{OC}_1 - \text{OC}_2,$$

where OC_1 and OC_2 are consecutive oscillatory circuits with the characteristic parameters

$$\begin{cases} \text{Case A: } [\omega_r]_{1,2} = \omega_0 \pm \Delta_A(\xi, \nu), \\ [\gamma_r]_{1,2} = \gamma_A(\xi, \nu), \\ \text{Case B: } [\omega_r]_{1,2} = \omega_0, \\ [\gamma_r]_1 = \gamma_{B,1}, \quad [\gamma_r]_2 = \gamma_{B,2}. \end{cases}$$

4. THE DURATION OF THE DESIRED SIGNAL IN THE OUTPUT OF A PHYSICALLY REALIZABLE WF

One of the assumptions of the RGA theory is that $S(j\omega) \propto G(j\omega)$, where $S(j\omega \longleftrightarrow s(t))$ is the spectrum of the desired signal $s(t)$. Within this approach the spectrum $S_w(j\omega)$ of the desired signal $s_w(t)$ in the output of a physically realizable WF with the transfer function (12) is determined by the following expression

$$s(t) \longleftrightarrow S_w(j\omega) \propto G(j\omega)H_w^{(r)}(j\omega) \propto [\Psi^+(j\omega)]^{-1}. \quad (14)$$

The last equation implies that the desired signal $s_w(t)$ can be considered as the momentum characteristics of the linear system $OC_1 - OC_2$. The duration τ^+ of the transient processes in such a system is determined by the following formula

$$\tau_A^+ \approx \gamma_A^{-1}(\xi, \nu) \text{ and } \tau_B^+ \approx \gamma_{B,2}^{-1}(\xi, \nu). \quad (15)$$

5. THE MAIN RESULTS AND CONCLUSIONS

Non-stationary WF. A physically realizable WF for resonant gravitational antennas containing a displacement converter is the optimal Kalman filter with a negative feedback. The parameters of the Kalman filter a determined by the matrices P and Q, which elements are solution of the dynamical equations (2) taking into dissipative forces: $K_i \longrightarrow K_i + pK_i$, $i = 1, 2$. The nonlinear dispersion equation (6) can be solved by analytical and numerical integration methods see [5]. As t increases the solution of the dispersion equation approaches the stationary solution, which is not dependent on the initial conditions. The duration of the transient processes in the Kalman filter output is determined by Eq. (15), where $\tau_{A,B}^+ \ll \tau^*$.

The stationary WF. The transfer function of the stationary physically realizable WF can be found from its gain frequency characteristics. One has to represent the spectral density $N(\omega)$ of the additive noise as a product of terms corresponding to the routes with non-negative imaginary parts (see Eq. (12)).

An adaptive processing of the information from resonant gravitational-wave antenna outputs. The spectral density of a stationary random process is usually estimated under the assumption that the measurement time is much longer than the correlation time. For gravitation-wave experiment this not always the case: the correlation length of the additive noise in the line channel output is $\tau^* \approx 1400$–2000 sec, while the measurement interval duration does not exceed the stationarity period $T_s \lesssim 2$ hours. Such a restriction can be significantly weakened if we apply statistical analysis to the random processes in the IF output (see above). The IF transfer function is determined by the parameters of the mechanical system. The additive noise in the IF output can be treated as a wideband random process, which correlation time τ_{IF} is determined by the width of the complex envelop $\tilde{x}(t)$ spectrum

$$\tau_{IF} \approx \pi/\Delta_\omega \ll \tau^*.$$

The inverse filtration preserves the complete information on the desired signal.

Information processing in the presence of non-Gaussian noises. The frequency-amplitude suppression of correlated non-Gaussian noises in the detection theory, see [7], is accomplished via the scheme WF–ILT–MF, where ILC is the inertialess converter with the optimal (according to the signal-to-noise ratio criterion) characteristics. Application of such a scheme in gravitational-wave experiments becomes worthwhile after the preliminary inverse filtering. Indeed, the WF transfer function is determined by the spectral density of the incoming additive noise.

The inverse filtration performs a preliminary decorrelation of mechanical noises, which ensures a high precision of the spectral density estimation for “short” observation intervals.

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