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Mathematical Model for Antisymmetric Solutions of *N*-particle Schrodinger Equation

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Abstract—We consider equations for a mathematical model system of fermions. Equations for spectrum are determined from the system of variational equations. The eigenvalues of the system of variational equations were defined for a particular solution.

Key words: quantum statistics, quantum model system, ultrasecondary quantization, variational equations, energy spectrum of excitations, Fermi particles.

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In the famous paper by academician N.N. Bogoliubov [1] the N-particle Hamiltonian with a weak interaction $V(|\mathbf{r}_i - \mathbf{r}_j|)$ between particles i and j was considered for the case of symmetric solutions. The key condition of N.N. Bogoliubov

$$\int V(r)d\mathbf{r}^3 > 0$$

implies that repulsion, on the average, dominates in the interaction. If, however,

$$\int V(r)d\mathbf{r}^3 < 0,$$

the system is unstable. In [2] the passage to the δ-function has been performed in the interaction potential, corresponding to the thermodynamic limit.

It is well known that He^3 and He^4 particles repulse each other if approached and attract mutually when moved away. Bellow we consider the case

$$\int V(r)d\mathbf{r}^3 = 0, \tag{1}$$

when, on the average, there is neither repulsion nor attraction and concern a antisymmetric case, which are known to correspond to He^3.

Let us consider a system of fermions located on a three-dimensional torus T with the side lengths L_1, L_2, and L_3. With the thermodynamic limit reasons in mind (see [2] for more details), we obtain the interaction potential having the property (1), namely,

$$V(x, y) = V_0\Delta_x\delta(x-y), \tag{2}$$

where x, y are the coordinates of particles, $\delta(x, y)$ is the Dirac delta function, Δ_x is the Laplace operator acting on the argument x.

In the limit as $N \longrightarrow \infty$, the asymptotic of the eigenvalue series for the system of N identical fermions is determined by the solutions of the following system of equations [3]:

$$\begin{gathered}
-\frac{\hbar^2}{2m}(\Delta_x - \Delta_y)G(x, y) \\
-\int dz(V(x, z) - V(y, z))\tilde{R}(x, z)R(z, y) = 0, \\
-\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)R(x, y) - \int dz(V(x, z)R(x, z)G(z, y) \\
+ V(y, z)R(z, y)G(z, x)) = \Omega R(x, y), \\
-\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)\tilde{R}(x, y) - \int dz(V(x, z)\tilde{R}(x, z)G(y, z) \\
+ V(y, z)\tilde{R}(z, y)G(x, z)) = \Omega\tilde{R}(x, y).
\end{gathered} \tag{3}$$

Here m is the particle mass, $\hbar$ is the Planck constant, Ω is a real number. The functions $G(x, y)$, $R(x, y)$, $\tilde{R}(x, y)$ are defined on $L_2(T^2)$. Moreover, in the fermionic case $R(x, y)$, $\tilde{R}(x, y)$ are antisymmetric with respect to the permutation of x and y.

Taking into account Eq.(2), we obtain

$$\begin{gathered}
-\frac{\hbar^2}{2m}(\Delta_x - \Delta_y)G(x, y) - V_0(\Delta_z(\tilde{R}(x, z)R(z, y))|_{z = x} \\
-\Delta_z(\tilde{R}(x, z)R(z, y))|_{z = y}) = 0.
\end{gathered}$$

$$\begin{gathered}
-\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)R(x, y) - V_0(\Delta_z(G(z, y)R(x, z))|_{z = x} \\
+\Delta_z(G(z, x)R(z, y))|_{z = y}) = \Omega R(x, y),
\end{gathered} \tag{4}$$

$$-\frac{\hbar^2}{2m}(\Delta_x+\Delta_y)\tilde{R}(x,y)-V_0(\Delta_z(G(y,z)\tilde{R}(x,z))\big|_{z=x}$$
$$+\Delta_z(G(x,z)\tilde{R}(z,y))\big|_{z=y})=\Omega\tilde{R}(x,y).$$

Equations (4) permits the substitution

$$G(x,y)=\frac{1}{L_1L_2^2}e^{-ik(x-y)}\sum_l G_l e^{il(x-y)},$$

$$\tilde{R}(x,y)=\frac{1}{L_1L_2^2}e^{-ik(x+y)}\sum_l \tilde{R}_l e^{il(x-y)}, \quad (5)$$

$$R(x,y)=\frac{1}{L_1L_2^2}e^{ik(x+y)}\sum_l R_l e^{il(x-y)},$$

where k, l are three-dimensional vectors of the form

$$2\pi\left(\frac{n_1}{L_1},\frac{n_2}{L_2},\frac{n_3}{L_2}\right), \quad (6)$$

and n_1, n_2, n_3 are integers. In view of the antisymmetry of $R(x, y)$, $\tilde{R}(x, y)$ the following relations should be hold

$$R_l=-R_{-l}, \quad \tilde{R}_l=-\tilde{R}_{-l}. \quad (7)$$

After substitution of Eqs. (5) into Eqs. (4) we get the equations

$$\tilde{\mathrm{r}}R_l-\mathrm{r}\tilde{R}_l=0,$$
$$\Omega\tilde{R}_l=\frac{\hbar^2}{m}(k^2+l^2)\tilde{R}_l+2V_0l\tilde{r}(G_l+G_{-l}), \quad (8)$$
$$\Omega R_l=\frac{\hbar^2}{m}(k^2+l^2)R_l+2V_0lr(G_l+G_{-l}),$$

where the vectors r, $\tilde{\mathrm{r}}$ are given by

$$\mathrm{r}=\frac{1}{L_1L_2^2}\sum_l lR_l, \quad \tilde{\mathrm{r}}=\frac{1}{L_1L_2^2}\sum_l l\tilde{R}_l. \quad (9)$$

The general solution of the system (8) has the form

$$R_l=-\frac{2V_0l\mathrm{r}(G_l+G_{-l})}{\hbar^2(k^2+l^2)/m-\Omega},$$
$$\tilde{R}_l=-\frac{2V_0l\tilde{\mathrm{r}}(G_l+G_{-l})}{\hbar^2(k^2+l^2)/m-\Omega}. \quad (10)$$

Substitution of Eq. (10) into Eq. (9) yields

$$\mathrm{r}=-\frac{4V_0}{L_1L_2^2}\sum_l (r,l)\frac{lG_l}{\hbar^2(k^2+l^2)/m-\Omega},$$
$$\tilde{\mathrm{r}}=-\frac{4V_0}{L_1L_2^2}\sum_l (\tilde{r},l)\frac{lG_l}{\hbar^2(k^2+l^2)/m-\Omega}, \quad (11)$$

where the brackets $(\cdot,\cdot)$ denote the scalar product of two vectors. Each of these equations represents a homogeneous system of three equations in the components of the vectors r and $\tilde{\mathrm{r}}$.

We note that in the one-dimensional case Eq. (11) transforms into the equality

$$-\frac{4V_0}{L_1L_2^2}\sum_l\frac{l^2G_l}{\hbar^2(k^2+l^2)/m-\Omega}=1.$$

Eqs. (8) can be written in the following form

$$[A_l, L_l]=0. \quad (12)$$

Here the matrices L_l and A_l have the form

$$L_l=\begin{pmatrix}\tilde{\varepsilon}_l & -V\tilde{\alpha}_l\\ V\alpha_l & -\tilde{\varepsilon}_{-l}\end{pmatrix}, \quad A_l=\begin{pmatrix}G_l & \tilde{R}_l\\ -R_l & -G_{-l}\end{pmatrix}, \quad (13)$$

where

$$\tilde{\varepsilon}_l=\frac{\hbar^2}{2m}(k_1-l)^2-\frac{\Omega}{2},$$
$$\alpha_l=2lr, \quad (14)$$
$$\tilde{\alpha}_l=2l\tilde{r},$$

and r, $\tilde{\mathrm{r}}$ are given by Eq. (9).

Let us consider solutions of Eq. (12) of the form

$$A_l=f(L_l). \quad (15)$$

The eigenvalues of the matrix L_l are

$$\tilde{\mu}_l^{\pm}=-\frac{\hbar^2}{m}kl\pm\mu_l,$$

where

$$\mu_l=\pm\sqrt{\varepsilon_l^2-V^2\tilde{\alpha}_l\alpha_l}.$$

For the matrices (13) we get from Eq. (15):

$$G_l=\frac{\mu_l+\varepsilon_l}{2\mu_l}f(\tilde{\mu}_l^+)+\frac{\mu_l-\varepsilon_l}{2\mu_l}f(\tilde{\mu}_l^-),$$
$$G_{-l}=-\frac{\mu_l-\varepsilon_l}{2\mu_l}f(\tilde{\mu}_l^+)-\frac{\mu_l+\varepsilon_l}{2\mu_l}f(\tilde{\mu}_l^-),$$
$$R_l=\frac{V\alpha_l}{2\mu_l}(f(\tilde{\mu}_l^-)-f(\tilde{\mu}_l^+)), \quad (16)$$
$$\tilde{R}_l=\frac{V\tilde{\alpha}_l}{2\mu_l}(f(\tilde{\mu}_l^-)-f(\tilde{\mu}_l^+)),$$

where

$$\varepsilon_l=\frac{\tilde{\varepsilon}_l+\tilde{\varepsilon}_{-l}}{2}=\frac{\hbar^2}{2m}(k_1^2+l^2)-\frac{\Omega}{2}.$$

Substitution of R_l, $\tilde{R}_l$ of the form (16) into Eq. (9) leads to the following equations

$$r = \frac{V_0}{2L_1L_2^2}\sum_l \frac{l\alpha_l}{\mu_l}(f(\tilde{\mu}_l^-) - f(\tilde{\mu}_l^+)),$$
$$\tilde{r} = \frac{V_0}{2L_1L_2^2}\sum_l \frac{l\tilde{\alpha}_l}{\mu_l}(f(\tilde{\mu}_l^-) - f(\tilde{\mu}_l^+)). \quad (17)$$

In view of Eqs. (16) the obtained equations agree with Eqs. (11). In the one-dimensional case these equations reduce to the condition

$$\frac{V_0}{L_1L_2^2}\sum_l \frac{l^2}{\mu_l}(f(\tilde{\mu}_l^-) - f(\tilde{\mu}_l^+)) = 1. \quad (18)$$

The function $f(x)$ is not completely arbitrary. From the antisymmetry condition (17) as well as from the equality

$$G_l - G_{-l} = -(G_{-l} - G_{-(-l)})$$

it follows that $f(x)$ should satisfy the equalities

$$f\left(-\frac{\hbar^2}{m}kl + \mu_l\right) - f\left(\frac{\hbar^2}{m}kl - \mu_l\right)$$
$$= f\left(\frac{\hbar^2}{m}kl + \mu_l\right) - f\left(\frac{\hbar^2}{m}kl - \mu_l\right),$$
$$f\left(-\frac{\hbar^2}{m}kl + \mu_l\right) + f\left(-\frac{\hbar^2}{m}kl - \mu_l\right)$$
$$= -f\left(\frac{\hbar^2}{m}kl + \mu_l\right) - f\left(\frac{\hbar^2}{m}kl - \mu_l\right).,$$

By adding and subtracting these equations, we get

$$f\left(-\frac{\hbar^2}{m}kl + \mu_l\right) = -f\left(\frac{\hbar^2}{m}kl - \mu_l\right),$$
$$f\left(-\frac{\hbar^2}{m}kl - \mu_l\right) = -f\left(\frac{\hbar^2}{m}kl + \mu_l\right).$$

Odd functions

$$f(\xi) = -f(-\xi)$$

satisfy these conditions for any value of k. In view of Eqs. (16), for $k = 0$, we have

$$G_l = G_{-l}.$$

Thus, the solutions (16) are determined by a odd function $f(\cdot)$ and by the parameters r, $\tilde{r}$ obeying Eqs. (18).

Let us consider the variational equations corresponding to Eq. (3):

$$-\lambda\delta G(x, y) = \frac{\hbar^2}{2m}(\Delta_x - \Delta_y)\delta G(x, y)$$
$$+ \int dz(V(x, z) - V(y, z))$$
$$\times(\tilde{R}(x, z)\delta R(z, y) + \delta\tilde{R}(x, z)R(z, y)),$$
$$(\Omega - \lambda)\delta R(x, y) = \left(-\frac{\hbar^2}{2m}(\Delta_x + \Delta_y)\right)\delta R(x, y)$$
$$-\int dz V(x, z)(R(x, z)\delta G(z, y) + \delta R(x, z)G(z, y)) \quad (19)$$
$$-\int dz V(y, z)(R(z, y)\delta G(z, x) + \delta R(z, y)G(z, x)),$$
$$(\Omega + \lambda)\delta\tilde{R}(x, y) = \left(-\frac{\hbar}{2m}(\Delta_x + \Delta_y)\right)\delta\tilde{R}(x, y)$$
$$-\int dz V(x, z)(\tilde{R}(x, z)\delta G(y, z) + \delta\tilde{R}(x, z)G(y, z))$$
$$-\int dz V(y, z)(\tilde{R}(z, y)\delta G(x, z) + \delta\tilde{R}(z, y)G(x, z)).$$

For the potential (2) we seek the solution of these equations in the form

$$\delta G(x, y) = \frac{1}{L_1L_2^2}e^{-ik_1(x-y)}\sum_{pq}\delta G_{pq}e^{ipx+iqy},$$
$$\delta\tilde{R}(x, y) = \frac{1}{L_1L_2^2}e^{-ik_1(x+y)}\sum_{pq}\delta\tilde{R}_l e^{ipx+iqy}, \quad (20)$$
$$\delta R(x, y) = \frac{1}{L_1L_2^2}e^{ik_1(x+y)}\sum_{pq}\delta R_{pq}e^{ipx+iqy}.$$

In view of antisymmetry of the functions $\delta R(x, y)$, $\delta\tilde{R}(x, y)$ the following relations should be hold

$$\delta R_{pq} = -\delta R_{qp}, \quad \delta\tilde{R}_{pq} = -\delta\tilde{R}_{qp}.$$

Substitution of Eq. (20) into Eq. (19) leads to the equations

$$(\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda)\delta G_{pq} - V_0\tilde{\alpha}_p\delta R_{pq} - V_0\alpha_q\delta\tilde{R}_{pq}$$
$$- V_0(p-q)(R_qB_{p+q} + \tilde{R}_pA_{p+q}) = 0,$$
$$(\tilde{\varepsilon}_{-p} + \tilde{\varepsilon}_{-q} + \lambda)\delta R_{pq} - V_0\alpha_p\delta G_{pq} + V_0\alpha_q\delta G_{pq}$$
$$+ V_0(p-q)(G_{-p} + G_{-q})A_{p+q} = 0, \quad (21)$$
$$(\tilde{\varepsilon}_p + \tilde{\varepsilon}_q - \lambda)\delta\tilde{R}_{pq} - V_0\tilde{\alpha}_p\delta G_{qp} + V_0\tilde{\alpha}_q\delta G_{pq}$$
$$+ V_0(p-q)(G_p + G_q)B_{p+q} = 0,$$

where

$$A_k = \frac{1}{L_1L_2^2}\sum_s s\delta R_{k-s,s}, \quad B_k = \frac{1}{L_1L_2^2}\sum_s s\delta\tilde{R}_{k-s,s}. \quad (22)$$

In Eq. (22) summation is performed over all vectors s of the form (6), and α_l, $\tilde{\alpha}_l$ are determined by Eqs. (14).

Eqs. (21), after substitution of δG_{pq}, δG_{qp}, can be rewritten as:

$$\delta G_{pq} = \frac{V_0}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda}(\tilde{\alpha}_p \delta R_{pq} + \alpha_q \delta \tilde{R}_{pq} + (p-q)R_q B_{p+q} + (p-q)\tilde{R}_p A_{p+q}),$$

$$\begin{aligned}
&(\tilde{\varepsilon}_{-p} + \tilde{\varepsilon}_{-q} + \lambda)\delta R_{pq} + V_0(p-q)(G_{-p} + G_{-q})A_{p+q} \\
&- \frac{V_0^2 \alpha_p}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda}(\tilde{\alpha}_p \delta R_{pq} + \alpha_q \delta \tilde{R}_{pq} + (p-q)R_q B_{p+q} + (p-q)\tilde{R}_p A_{p+q}) \\
&- \frac{V_0^2 \alpha_q}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-q} - \lambda}(\tilde{\alpha}_q \delta R_{pq} + \alpha_p \delta \tilde{R}_{pq} + (p-q)R_p B_{p+q} + (p-q)\tilde{R}_q A_{p+q}) = 0, \\
&(\tilde{\varepsilon}_p + \tilde{\varepsilon}_q - \lambda)\delta \tilde{R}_{pq} + V_0(p-q)(G_p + G_q)B_{p+q} \\
&+ \frac{V_0^2 \tilde{\alpha}_q}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda}(\tilde{\alpha}_p \delta R_{pq} + \alpha_p \delta \tilde{R}_{pq} + (p-q)R_q B_{p+q} + (p-q)\tilde{R}_p A_{p+q}) \\
&+ \frac{V_0^2 \tilde{\alpha}_p}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda}(\tilde{\alpha}_q \delta R_{pq} + \alpha_p \delta \tilde{R}_{pq} + (p-q)R_p B_{p+q} + (p-q)\tilde{R}_q A_{p+q}) = 0.
\end{aligned} \tag{23}$$

From Eqs. (23) it follows that δR_{pq}, $\delta \tilde{R}_{pq}$ are expressed through A_{p+q}, B_{p+q} by the following way:

$$\begin{aligned}
\delta R_{pq} &= \zeta_{p,q} A_{p+q} + \sigma_{p,q} B_{p+q}, \\
\delta \tilde{R}_{pq} &= \eta_{p,q} A_{p+q} + \xi_{p,q} B_{p+q},
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
\zeta_{p,q} &= \frac{M_{4,p,q}}{D_{p,q}} V_0(p-q)\left(\frac{V_0 \alpha_p \tilde{R}_p}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda} + \frac{V_0 \alpha_q \tilde{R}_q}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda} - G_{-p} - G_{-q}\right) \\
&+ \frac{M_{2,p,q}}{D_{p,q}} V_0^2(p-q)\left(\frac{\tilde{\alpha}_q \tilde{R}_p}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda} + \frac{\tilde{\alpha}_p \tilde{R}_q}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda}\right), \\
\sigma_{p,q} &= \frac{M_{4,p,q}}{D_{p,q}} V_0^2(p-q)\left(\frac{\alpha_p R_q}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda} + \frac{\alpha_q R_p}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda}\right) \\
&+ \frac{M_{2,p,q}}{D_{p,q}} V_0(p-q)\left(\frac{V_0 \tilde{\alpha}_q R_q}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda} + \frac{V_0 \tilde{\alpha}_p R_p}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda} + G_p + G_q\right), \\
\eta_{p,q} &= -\frac{M_{1,p,q}}{D_{p,q}} V_0^2(p-q)\left(\frac{\tilde{\alpha}_q \tilde{R}_p}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda} + \frac{\tilde{\alpha}_p \tilde{R}_q}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda}\right) \\
&- \frac{M_{3,p,q}}{D_{p,q}} V_0(p-q)\left(\frac{V_0 \alpha_p \tilde{R}_p}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda} + \frac{V_0 \alpha_q \tilde{R}_q}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda} - G_{-p} - G_{-q}\right), \\
\xi_{p,q} &= -\frac{M_{1,p,q}}{D_{p,q}} V_0(p-q)\left(\frac{V_0 \tilde{\alpha}_q R_q}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda} + \frac{V_0 \tilde{\alpha}_p R_p}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda} + G_p + G_q\right) \\
&- \frac{M_{3,p,q}}{D_{p,q}} V_0^2(p-q)\left(\frac{\alpha_p R_q}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda} + \frac{\alpha_q R_p}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda}\right),
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
D_{p,q} &= M_{1,p,q} M_{4,p,q} - M_{2,p,q} M_{3,p,q}, \\
M_{1,p,q} &= \tilde{\varepsilon}_{-p} + \tilde{\varepsilon}_{-q} + \lambda - \frac{V_0^2 \alpha_p \tilde{\alpha}_p}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda} - \frac{V_0^2 \alpha_q \tilde{\alpha}_q}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda}, \\
M_{2,p,q} &= -\frac{V_0^2 \alpha_p \alpha_q}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda} - \frac{V_0^2 \alpha_p \alpha_q}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda}, \\
M_{3,p,q} &= \frac{V_0^2 \tilde{\alpha}_p \tilde{\alpha}_q}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda} + \frac{V_0^2 \tilde{\alpha}_p \tilde{\alpha}_q}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda}, \\
M_{4,p,q} &= \tilde{\varepsilon}_p + \tilde{\varepsilon}_q - \lambda + \frac{V_0^2 \alpha_q \tilde{\alpha}_q}{\tilde{\varepsilon}_p - \tilde{\varepsilon}_{-q} - \lambda} + \frac{V_0^2 \alpha_p \tilde{\alpha}_p}{\tilde{\varepsilon}_q - \tilde{\varepsilon}_{-p} - \lambda}.
\end{aligned} \tag{26}$$

Substitution of Eq. (24) into (22) leads to the homogeneous system of linear equations in the components of the vectors A_p, B_p:

$$\begin{aligned}
A_p &= \frac{1}{L_1 L_2^2} \sum_s s((\zeta_{p-s,s} A_p) + (\sigma_{p-s,s} B_p)), \\
B_p &= \frac{1}{L_1 L_2^2} \sum_s s((\eta_{p-s,s} A_p) + (\xi_{p-s,s} B_p)),
\end{aligned} \tag{27}$$

where brackets (,) denote the scalar product of two vectors.

The system of linear equations (21) has the solutions $\delta G^{(l)}_{p,q}$, $\delta R^{(l)}_{p,q}$, $\delta \tilde{R}^{(l)}_{p,q}$ possessing the properties

$$\delta G^{(l)}_{p,q} = \delta R^{(l)}_{p,q} = \delta \tilde{R}^{(l)}_{p,q} = 0 \text{ at } p+q \neq l. \quad (28)$$

For the solutions (28) the relations

$$A_k = 0, \quad B_k = 0 \quad \text{at} \quad k \neq l$$

are valid. Which means that for these solutions the system of an infinite number of equations (27) reduces to the homogeneous system of two vector equations at $k = l$. The equality of the corresponding determinant to zero represents the equation for energy spectrum.

PARTICULAR SOLUTION

The system of equations (8) has the solution

$$G_p = \delta_{p,0}, \quad R_p = \tilde{R}_p = 0.$$

Substituting this solution into (21) and taking into account Eq. (22)

$$A_l = \frac{1}{L_1 L_2^2} l \delta R_{0,l}, \quad B_l = \frac{1}{L_1 L_2^2} l \delta \tilde{R}_{0,l}$$

we get the homogenous system of linear equations in A_{1l} and B_{1l}. This system can be written in the form

$$Q_l X_l = 0,$$

where

$$Q_l = \begin{pmatrix} \tilde{\varepsilon}_{-l} + \tilde{\varepsilon}_0 + \lambda - \dfrac{V_0 l^2}{L_1 L_2^2} & 0 \\ 0 & \tilde{\varepsilon}_l + \tilde{\varepsilon}_0 - \lambda - \dfrac{V_0 l^2}{L_1 L_2^2} \end{pmatrix},$$

$$X_l = \begin{pmatrix} A_{1,l} \\ B_{1,l} \end{pmatrix}.$$

By equating the determinant of the matrix Q_l to zero

$$\det Q_l = \left(\tilde{\varepsilon}_{-l} + \tilde{\varepsilon}_0 + \lambda - \frac{V_0 l^2}{L_1 L_2^2} \right) \times \left(\tilde{\varepsilon}_l + \tilde{\varepsilon}_0 - \lambda - \frac{V_0 l^2}{L_1 L_2^2} \right) = 0,$$

we find the spectrum of quasiparticles:

$$\lambda_{1,l} = \tilde{\varepsilon}_l + \tilde{\varepsilon}_0 - \frac{V_0 l^2}{L_1 L_2^2}, \quad \lambda_{2,l} = -\tilde{\varepsilon}_{-l} - \tilde{\varepsilon}_0 + \frac{V_0 l^2}{L_1 L_2^2}.$$

Taking into account that

$$\tilde{\varepsilon}_l = \frac{\hbar^2}{2m}(k-l)^2 - \frac{\Omega}{2},$$

we get

$$\lambda_{1,l} = \frac{\hbar^2 l(l-2k)}{2m} + \frac{\hbar^2 k^2}{m} - \Omega - \frac{V_0 l^2}{L_1 L_2^2},$$

$$\lambda_{2,l} = -\frac{\hbar^2 l(l+2k)}{2m} - \frac{\hbar^2 k^2}{m} + \Omega + \frac{V_0 l^2}{L_1 L_2^2}.$$

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REFERENCES

1. N. N. Bogolyubov, Izv. Akad. Nauk SSSR, Ser. Fiz. **11**, 77 (1947); J. of Phys. **9**, 23 (1947).
2. E. M. Lifshits and L. P. Pitaevskii, *Statistical Physics* (Fizmatlit, Moscow, 2000) [in Russian].
3. V. P. Maslov, Russ. J. Math. Phys. **9** (4), 437 (2002).
4. V. P. Maslov and O. Yu. Shvedov, *Complex Germ Method in the Many-Particle Problem and Quantum Field Theory* (URSS, Moscow, 2000) [in Russian].
5. V. P. Maslov, Teor. Mat. Fiz. **143** (3), 307 (2005) [Theor. Math. Phys. **143**, 741 (2005)].