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Cooling Kinetics of a Granular Gas of Viscoelastic Particles

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Abstract—The evolution of a granular gas of viscoelastic particles in the homogeneous cooling state is studied. The velocity distribution function of granular particles and the time dependence of the mean kinetic energy of particles (granular temperature) are found. The noticeable deviation of the distribution function from the Maxwell distribution and its non-monotonous evolution are established. The perturbation theory with respect to the small dispersion parameter is elaborated and the analytical expressions for the asymptotic time dependence of the velocity distribution function and the granular gas temperature are derived.

Key words: kinetic theory of gases, granular gas, dissipative gas, velocity distribution function, restitution coefficient, viscoelastic particles.

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INTRODUCTION

Granular media are the total of microscopic particles with the dimension from a part of a micrometer to meters. They are widespread in nature. Powders, sand, and gravel may serve as examples [1]. Friction in these systems results in the appearance of a series of unusual properties: under weak external impacts they keep their form, in a similar manner to solids. However, under a considerable load, fluid-like flows may arise in granular media [2]. Thin granular systems are called granular gases [3–7]. Astrophysical objects such as dust clouds and planetary rings, including Saturn's rings, are granular gases as well [5, 6].

During the collision of particles, a part of their kinetic energy transforms into the energy of the heat motion of molecules composing a granule [3]. The value of this loss may be characterized by the restitution coefficient ε [3]:

$$\varepsilon = \frac{|(v'_{12}e)|}{|(v_{12}e)|}.$$

Here v_{12} and v'_{12} are relative particle velocities before and after a collision, respectively and **e** is the unit vector connecting particle centers. For the sake of simplicity, it is assumed that all particles are spherical.

To simplify the calculations, the restitution coefficient ε is often assumed to be a constant, not depending on the relative velocity of the colliding granules. However, this is in contradiction with the experimental data [9–11] and is not confirmed by the theoretical analysis [12]. By integrating Newton equations for colliding viscoelastic particles with the force of their interaction depending on the distance between their centers and the relative particle velocities according to the viscoelastic interaction law [3, 13], it is possible to obtain the following dependence of the restitution coefficient on the relative particle velocity at the collision [3, 14]:

$$\varepsilon = C_0 - C_1\delta'(t)(c_{12}e)^{\frac{1}{5}} + C_2\delta'^2(t)(c_{12}e)^{\frac{2}{5}} \mp \ldots, \quad (1)$$

where $C_0 = 1$, C_1 and C_2 are numerical coefficients on the order of unity [3], $\delta'(t) = \delta(u(t))^{1/10}$, $u(t) = T(t)/T(0)$ is the relative gas temperature, δ is a small parameter characterizing the dissipation of the particle energy (its dependence on the material parameters may be found in [3]), $c_{12} = v_{12}/v_T$ is the dimensionless relative velocity, $v_T = 2T(t)/m$ is the mean-square velocity, m is the particle mass. The granular gas temperature $T(t)$ is the mean kinetic energy of the translational motion of a granule:

$$\frac{3}{2}nT = \int dv \frac{mv^2}{2} f(v, t),$$

where n is the gas concentration and $f(v, t)$ is the velocity distribution function. Below we use its scaling form

$$f(v, t) = \frac{n}{v_T^3(t)}\tilde{f}(c, t).$$

Due to the dissipative character of the interparticle interactions the gas relative velocity distribution function $f(c, t)$ differs noticeably from the Maxwell distribution $\varphi(c) = \pi^{-3/2}\exp(-c^2)$ typical for common molecular gases. If dissipation is not very large, this deviation may be described by the Sonine polynomial series expansion [15, 16]:

$$\tilde{f}(c,t) = \varphi(c)(1+\sum a_p(t)S_p(c^2)). \tag{2}$$

It follows from the definition of temperature that the coefficient of the first polynomial $a_1(t) = 0$ [3]. In [17] it was supposed that the behavior of the coefficients in the expansion Eq. (2) follows the power dependence:

$$a_k \sim \lambda^k, \tag{3}$$

where λ is a small parameter, Its physical nature was not discussed in [17]. However, in [18] it was shown that for granular gases with a constant restitution coefficient the coefficient a_3 is comparable in its order of magnitude with a_2, which makes the hypothesis in Eq. (3) doubtful. As noted above, the supposition that the restitution coefficient is constant contradicts the experimental data. Therefore the aim of the present work is the analysis of the distribution function and temperature of the granular gas of viscoelastic particles with the restitution coefficient Eq. (1) and checking if the hypothesis Eq. (3) holds for the given system. The model of viscoelastic particles has an advantage of the explicit dependence of the studying quantities on the small dissipation parameter δ.

Calculation of a_2 and a_3 coefficients.

The Boltzmann equation for the gas velocity distribution function is as follows [19]:

$$\frac{\partial f(v,t)}{\partial t} = g_2(\sigma)I(f,f), \tag{4}$$

where $I(f, f)$ is the collision integral, $g_2(\sigma)$ is the pair correlation function taking into account the increasing collision frequency of particles due to its excluded volume and the corresponding spatial correlations [20] and σ is the particle diameter. For relative velocities it is convenient to introduce the dimensionless collision integral $\tilde{I}(\tilde{f}, \tilde{f})$ related to the conventional collision integral $I(f, f) = \sigma^2 n^2 v_T^{-2}\tilde{I}(\tilde{f},\tilde{f})$ [3] by:

$$I(\tilde{f},\tilde{f}) = \int d\tilde{n}_2\int de\Theta(-\tilde{n}_{12}e)|\tilde{n}_{12}e|$$
$$\times\left(\frac{1}{\varepsilon^2}\tilde{f}(\tilde{n}_1,t)\tilde{f}(\tilde{n}_2,t)-\tilde{f}(\tilde{n}_1,t)\tilde{f}(\tilde{n}_2,t)\right),$$

where $\Theta(x) = 1$ at $x > 0$ and $\Theta(x) = 0$ at $x < 0$.

By introducing the momenta of the dimensionless collision integral

$$\mu_p = -\int dc_1 c_1^p\tilde{I}(\tilde{f},\tilde{f})$$

and substituting the velocity distribution function Eq. (2) into the Boltzmann equation Eq. (4), one obtains the following system of equations [3]:

$$\begin{cases}\dfrac{\mu_2}{3}\left(3+c_1\dfrac{\partial}{\partial c_1}\right)\tilde{f}(c_1t)+B^{-1}\dfrac{\partial}{\partial t}\tilde{f}(c_1,t) = \tilde{I}(\tilde{f},\tilde{f}),\\ \dfrac{dT}{dt} = -\dfrac{2}{3}BT\mu_2.\end{cases}$$

Here $B = \sqrt{u(t)/(8\pi)}\tau_c^{-1}(0)$, where we introduced $\tau_c^{-1} = 4\sqrt{\pi}\,g_2(\sigma)\sigma^2 n\sqrt{T(0)/m}$ as the mean collision time in a gas [20].

By introducing the dimensionless time $\tau = t/\tau_c$, multiplying both parts of Eq. (4) by c_1^p and integrating over c_1, it is easy to obtain a system of equations to find the Sonine polynomial coefficients a_2, a_3 and relative temperature u:

$$\begin{cases}\dfrac{du}{d\tau} = -\dfrac{\sqrt{2}\mu_2}{8\sqrt{\pi}}u^{\frac{3}{2}},\\ \dfrac{da_2}{d\tau} = \dfrac{\sqrt{2}}{3\sqrt{\pi}}\mu_2(1+a_2)\sqrt{u}-\dfrac{\sqrt{2}}{15\sqrt{\pi}}\mu_4\sqrt{u},\\ \dfrac{da_3}{d\tau} = \dfrac{\sqrt{u}}{\sqrt{2\pi}}\mu_2(1-a_2+a_3)-\dfrac{\sqrt{2}}{5\sqrt{\pi}}\mu_4\sqrt{u}+\dfrac{2\sqrt{2u}}{105\sqrt{\pi}}\mu_6,\\ \mu_2 = (6.49+0.05a_2^2+1.56a_2+0.10a_3\\ +0.01a_3^2+0.04a_2a_3)\delta'(t)\\ +(-0.14a_3-0.01a_3^2-0.07a_2^2-9.29\\ -0.05a_2a_3-2.76a_2)\delta'^2(t)+O(\delta'^3(t)),\\ \mu_4 = (0.05a_3^2-2.51a_3+10.03a_2\\ +0.31a_2^2+0.16a_2a_3)\\ +(-6.50a_3+36.32-0.29a_2^2-0.14a_2a_3\\ -0.01a_3^2+46.85a_2)\delta'(t)\\ +(0.03a_3^2-100.66a_2+0.59a_2^2+0.25a_2a_3\\ +16.09a_3-71.50)\delta'^2(t)+O(\delta'^3(t)),\\ \mu_6 = (112.80a_2-84.60a_3-0.82a_2a_3\\ +0.02a_3^2-4.93a_2^2)\\ +(16.75a_2^2-245.02a_3+633.78a_2\\ +209.94+0.26a_3^2+2.23a_2a_3)\delta'(t)\\ +(-1718.36a_2-48.53a_2^2-0.45a_3^2\\ -525.04+717.54a_3-4.85a_2a_3)\delta'^2(t)\\ +O(\delta'^3(t)).\end{cases} \tag{5}$$

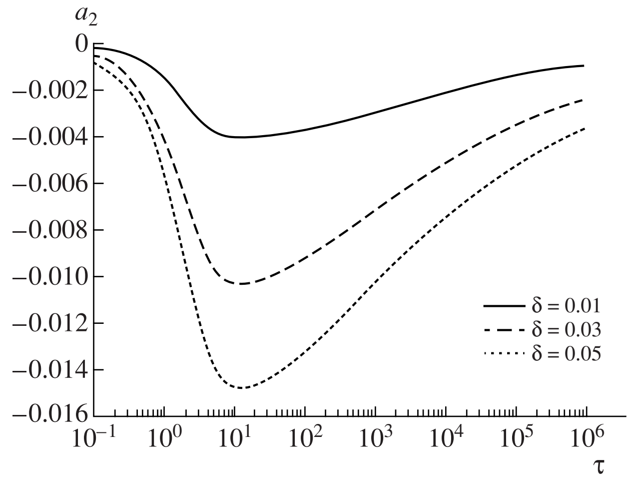

Fig. 1. Second coefficient in the Sonine polynomial expansion of the velocity distribution function $a_2(\tau)$ at $\delta = 0.01$, 0.03, and 0.05.

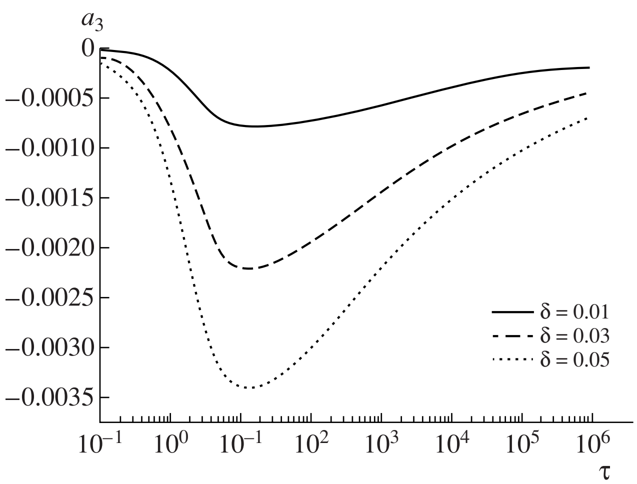

Fig. 2. Third coefficient in the Sonine polynomial expansion of the velocity distribution function $a_3(\tau)$ at $\delta = 0.01$, 0.03, and 0.05.

In the system of Eqs (5), for short, the numerical coefficients are expressed as simple fractions only in the first two equations, with those in the following equations expressed as decimal fractions. The analytical solution of the system of Eqs. (5) may be obtained by the perturbation theory with the series expansion of $u(\tau)$, $a_2(\tau)$ and $a_3(\tau)$ with respect to a small parameter δ: $u = u_0 + \delta u_1 + \ldots$, $a_2 = a_{20} + \delta a_{21} + \ldots$, $a_3 = a_{30} + \delta a_{31} + \ldots$. In the zero approximation with respect to δ, the solution of the system Eqs. (5) is as follows:

$$u_0(\tau) = \left(1 + \frac{\tau}{\tau_0}\right)^{-\frac{5}{3}}$$

$$a_{20}(\tau) = A_1 \exp\left(\frac{-149 + \sqrt{2041}}{210}\tau\right) + A_2 \exp\left(\frac{-149 - \sqrt{2041}}{210}\tau\right)$$

$$a_{30}(\tau) = A_1 \frac{-37 + \sqrt{2041}}{28} \exp\left(\frac{-149 + \sqrt{2041}}{210}\tau\right) + A_2 \frac{-37 - \sqrt{2041}}{28} \exp\left(\frac{-149 - \sqrt{2041}}{210}\tau\right),$$

where $\tau_0^{-1} = 0.56\delta$, A_1 and A_2 are the arbitrary constants depending on the initial conditions. In the case $a_{20}(0) = 0$ and $a_{30}(0) = 0$ the coefficients in the Sonine polynomial expansion of the distribution function in the zero approximation with respect to δ are zero in the total time interval: $a_{20}(\tau) = 0$ and $a_{30}(\tau) = 0$.

In the first approximation, $a_2^1 = a_{20} + a_{21}\delta$ and $a_3^1 = a_{30} + a_{31}\delta$, and the system Eqs. (5) is as follows:

$$\begin{cases} \dfrac{du_1}{d\tau} = -2.67\tau_0^{-1} u_0^{\frac{3}{5}} u_1 + 1.67\tau_0^{-1} u_0^{\frac{17}{10}} q_1 \\ - u_0^{\frac{8}{5}} \tau_0^{-1} (0.40 a_{21} + 0.03 a_{31}), \\ \dfrac{da_{21}}{d\tau} = -0.22 u_0^{\frac{3}{5}} + \sqrt{u_0}(-0.53 a_{21} + 0.13 a_{31}), \\ \dfrac{da_{31}}{d\tau} = -0.02 u_0^{\frac{3}{5}} + \sqrt{u_0}(0.11 a_{21} + 0.89 a_{31}). \end{cases} \quad (6)$$

By solving the system of Eqs. (6) at $\tau \longrightarrow \infty$, the following asymptotic expressions for the relative temperature $u(\tau)$ and coefficients $a_2(\tau)$ and $a_3(\tau)$ in the linear approximation with respect to δ:

$$u^1(\tau) = \left(\frac{\tau}{\tau_0}\right)^{-\frac{5}{3}} + 2.56\delta\left(\frac{\tau}{\tau_0}\right)^{-\frac{11}{6}} \quad (7)$$

$$a_2^1(\tau) = -0.44\delta(\tau/\tau_0)^{-\frac{1}{6}}, \quad (8)$$

$$a_3^1(\tau) = -0.08\delta(\tau/\tau_0)^{-\frac{1}{6}}. \quad (9)$$

may be obtained.

The second Eq. (8) and third Eq. (9) coefficients have the same asymptotic time dependence and the same order of smallness with respect to the dissipation parameter δ. Thus, the hypothesis given in [17], does not hold. The ratio of coefficients obtained in the asymptotic limit $\tau \longrightarrow \infty$ for small δ is $a_3^1/a_2^1 \approx 0.18$. It exceeds noticeably the conventional error of the theory of the homogeneous cooling of the granular gas of par-

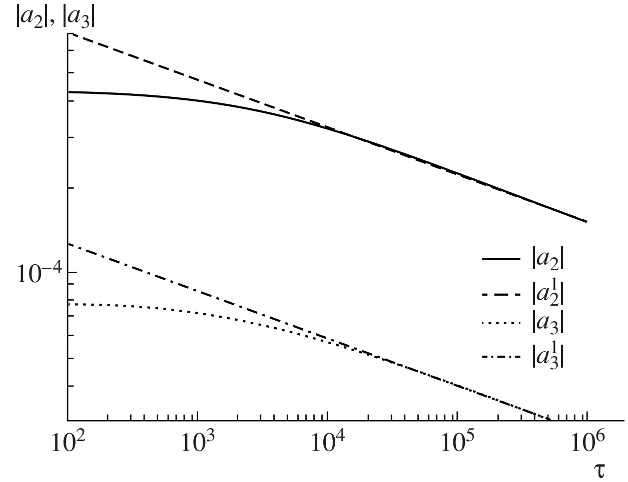

Fig. 3. Moduli of the second and third coefficients in the Sonine polynomial expansion of the velocity distribution function $|a_2(\tau)|$ and $|a_3(\tau)|$ at $\delta = 0.001$ and their linear approximation $|a_2^1(\tau)|$ and $|a_3^1(\tau)|$ in the limit $\tau \longrightarrow \infty$.

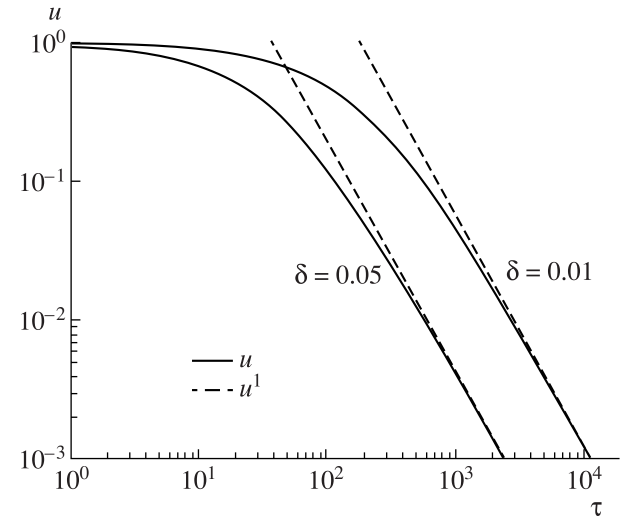

Fig. 4. Relative temperature $u(\tau)$ at $\delta = 0.01$ and 0.05. Dotted line follows the asymptotic $|u^1(\tau)|$ at $\tau \longrightarrow \infty$ Eq. (7).

ticles with the constant cooling coefficient (not taking into account a_3), which agrees with the computer simulation results with the accuracy of 2–3% [3, 21]. Thus, the third coefficient should be taken into account as well. At large δ when the linear approximation is violated, the ratio of the second coefficient to the third coefficient may increase [18]. The exact solution of the system Eq. (5) was obtained numerically at different values of the small dissipation parameter δ. It was supposed that at the initial time moment the system (5) obeys the Maxwell distribution, i.e., $a_2(0) = (0)$ and $a_3(0) = 0$. The time dependences of the second and third coefficients in the Sonine polynomial expansion $a_2(\tau)$ and $a_3(\tau)$ are given in Figs. 1 and 2. Figures show that initially $a_2(\tau)$ and $a_3(\tau)$ are negative and decrease with time reaching their minimum. The larger the parameter δ, the larger the value of $|a_2(\tau)|$ and $|a_3(\tau)|$. Then they increase tending to zero at $\tau \longrightarrow \infty$. Thus, the velocity distribution function tends to the Maxwell distribution. It should be noted that at very large times the behavior of the system becomes dramatically different because clusters and vortices are formed [3]. In the present paper we consider only the initial stage of the gas evolution called the "homogeneous cooling state" and suppose that the system remains in this regime. Figure 3 shows the comparison of the moduli of the second and third coefficients in the Sonine polynomial expansion of the velocity distribution function $|a_2(\tau)|$ and $|a_3(\tau)|$ obtained by the numerical solution of the system Eqs. (5) at sufficiently small value of the parameter $\delta = 0.001$ and moduli of the analytical solutions $|a_2^1(\tau)|$ Eq. (8) and $|a_3^1(\tau)|$ Eq. (9) obtained in the linear approximation with respect to δ at $\tau \longrightarrow \infty$. The asymptotics $|a_2^1(\tau)|$ and $|a_3^1(\tau)|$ agree well with the numerical values at $\tau >$ 10000. The τ dependence of the relative temperature u is shown in Fig. 4. The larger the parameter δ characterizing the dissipation, the lower the temperature and the higher the cooling velocity. The asymptotic u_1 obtained in the linear approximation with respect to δ at $\tau \longrightarrow \infty$ (Eq. (7) is shown with dashed lines in Fig. 4. It agrees well with the exact solution $u(\tau)$ after some time. The smaller δ is, the larger this time value becomes.

CONCLUSION

In the present work the evolution of the granular temperature and the velocity distribution function was investigated in the framework of the model of a granular gas of viscoelastic particles (with velocity-dependent restitution coefficients). This model is most close to reality. To calculate the velocity distribution function, the Sonine orthogonal polynomial expansion was used and the second and third coefficients of this expansion were obtained. It is interesting to note that the velocity distribution function relaxes to the Maxwell distribution at the same time scale as that for the change of the gas temperature itself. This is what radically differentiates the evolution of a gas of viscoelastic particles from that of a granular gas with a constant restitution coefficient, in which the distribution scaling function relaxes to the stationary state during a period on the order of several collisions but the characteristic time of gas cooling may be much longer [3, 21]. The third coefficient is of the same order of smallness with respect to the dissipation parameter as the second coefficient. Thus, it is necessary to take it into account for the adequate description of a granular gas.

REFERENCES

1. H. Hinrichsen and D. E. Wolf, *The Physics of Granular Media* (Wiley, Berlin 2004).
2. H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Rev. Mod. Phys. **68**, 1259 (1996).
3. N. V. Brilliantov and T. Pöschel, *Kinetic Theory of Granular Gases* (Oxford University Press, Oxford, 2004).
4. A. Barrat, E. Trizac, and M. H. Ernst, Phys. Cond. Matter **17**, 2429 (2005).
5. T. Pöschel and S. Luding, *Granular Gases* 564: *Lecture Notes in Physics* (Springer, Berlin, 2001).
6. T. Poeschel and N. V. Brilliantov, *Granular Gas Dynamics* Vol. 624: *Lecture Notes in Physics* (Springer, Berlin, 2003).
7. I. Goldhirsch, Ann. Rev. Fluid Mech. **35**, 267 (2003).
8. R. Greenberg and A. A. Brahic, *Planetary Rings* (Arizona University Press, Tucson, 1984).
9. W. Goldsmit, *The Theory and Physocal Behavior of Colliding Solids* (Arnold, London, 1960).
10. F. G. Bridges, A. Hatzes, and D. N. C. Lin, Nature **309**, 333 (1984).
11. G. Kuwabara and K. Kono, J. Appl. Phys. Part 1 **26**, 1230 (1987).
12. R. Ramirez, T. Pöschel and N. V. Brilliantov, Phys. Rev. E **60**, 4465 (1999).
13. N. V. Brilliantov, F. Spahn, J.-M. Hertsch, and T. Poeschel, Phys. Rev. E **53**, 5382 (1996).
14. T. Schwager and T. Pöschel, Phys. Rev. E **57**, 650 (1998).
15. A. Goldshtein and M. Shapiro, J. Fluid Mech. **282**, 75 (1995).
16. T. P. C. van Noije and M. H. Ernst, Granular Mater. **1**, 57 (1998).
17. M. Huthmann, J. A. Orza, and R. Brito, Granular Mater. **2**, 189 (2000).
18. N. V. Brilliantov and T. Pöschel, Europhys. Lett. **74**, 424 (2006).
19. E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics* (Moscow, 1979) [in Russian].
20. N. V. Brilliantov and O. P. Revokatov, *Molecular Dynamics of Disordered Media* (Moscow, 1996) [In Russian].
21. R. Brito and M. H. Ernst, Europhys. Lett. **43**, 497 (1998).