ISSN 0027-1349, Moscow University Physics Bulletin, 2009, Vol. 64, No. 2, pp. 195–201. © Allerton Press, Inc., 2009.
Original Russian Text © S.A. Gasanov, 2009, published in Vestnik Moskovskogo Universiteta. Fizika, 2009, No. 2, pp. 86–91.

Periodic Motion of a Star inside an Elliptical Galaxy with a Variable Mass

S. A. Gasanov

Sternberg State Astronomical Institute, Universitetskii pr. 13, Moscow, 119991 Russia
e-mail: gasanov@sai.msu.ru

Received August 19, 2008

Abstract—Periodic solutions are constructed to the problem of a star's motion inside a layered inhomogeneous rotating elliptical galaxy with a variable mass using the Lyapunov and the small parameter methods. The stability of periodic solutions in the Lyapunov sense is studied.

Key words: celestial mechanics, stellar dynamics and kinematics, extragalactic objects and systems.

DOI: 10.3103/S0027134909020209

1. STATEMENT OF THE PROBLEM

The problem of the spatial motion of a star inside a layered inhomogeneous rotating elliptical galaxy E with a variable mass was considered in [1]. The laws of time variation of density of ellipsoidal layers are assumed to differ from each other with a constant factor. It was shown in [2] that a more general case can be considered in which the density at the center of the galaxy O changes according to the law $\rho_0(t)$, and in the following layers, according to the laws $\rho_k(t)$ $k = 1, 2, \ldots$). The total density of the galaxy E in this case is represented in the form $\rho(t) = \rho_0(t) - \varepsilon\rho_1(t) - \varepsilon^2\rho_2(t) - \ldots - \varepsilon^k\rho_k(t)$, where ε is a dimensionless small parameter characterizing the density distribution.

Let us assume that an elliptical galaxy with a variable mass is bounded by the ellipsoidal surface with the half-axes $\bar{a}$, $\bar{b}$, and $\bar{c}$ ($\bar{a} \geq \bar{b} \geq \bar{c}$), has the total density $\rho(t)$, and rotates with the angular velocity $\Omega = \Omega(t)$ about the polar axis Oz. The density of each ellipsoidal layer in this case is a function of time. Let us consider terms up to ε^2 inclusive in the expression for the galaxy density $\rho(t)$. We also assume that the star S moves under the action of the attractive force of the galaxy only.

The coordinates x, y, z of the star S are related to the rectangular coordinate system with the point of origin at the center of the elliptical galaxy O and the axes directed along the corresponding main axes of inertia of this galaxy. The equations of motion of the star inside the gravitating galaxy E in this coordinate system represent the system of equations with the variable coefficients,

$$\frac{d^2x}{dt^2} - 2\Omega\frac{dy}{dt} - \Omega^2 x - \frac{d\Omega}{dt}y = \frac{\partial U}{\partial x},$$
$$\frac{d^2y}{dt^2} + 2\Omega\frac{dx}{dt} - \Omega^2 y + \frac{d\Omega}{dt}x = \frac{\partial U}{\partial y}, \qquad (1)$$
$$\frac{d^2z}{dt^2} = \frac{\partial U}{\partial z}.$$

In this case, the force function U is determined by the series in powers of the parameter ε,

$$U = V_0 + R,$$

where

$$V_0 = -\frac{\rho_0(t)}{2}(V_{01}x^2 + V_{02}y^2 + V_{03}z^2 - 2V_{00}),$$

$$\begin{aligned} R = {} & \frac{\varepsilon}{4}\rho_1(t)(R_1x^4 + R_2y^4 + R_3z^4 \\ & + 2R_4x^2y^2 + 2R_5x^2z^2 + 2R_6y^2z^2) \\ & + \frac{\varepsilon^2}{6}\rho_2(t)(S_1x^6 + S_2y^6 + S_3z^6 + 3S_4x^2y^4 \\ & + 3S_5x^4y^2 + 3S_6x^2z^4 + 3S_7x^4z^2 \\ & + 6S_8x^2y^2z^2 + 3S_9y^4z^2 + 3S_{10}y^2z^4) + \ldots. \end{aligned} \qquad (2)$$

In expansion (2) of the perturbing function R terms up to ε^2 inclusive are taken into account. For $\varepsilon = 0$ the force function $U = V_0$ represents the attractive potential of a homogeneous ellipsoid with a variable mass. A layered inhomogeneous ellipsoid with a variable mass corresponds to a nonzero ε. All coefficients $V_{0i} > 0$ ($i = 0, 1, 2, 3$) and $R_k > 0$ ($k = 1, 2, \ldots, 6$) were given in [3], and $S_n > 0$ ($n = 1, 2, \ldots, 10$), in [4]. These coefficients contain the gravitational constant G and represent fourth order polynomials with respect to the second eccentricities λ and μ, which are assumed to be small parameters and are connected with the half-axes $\bar{a}$, $\bar{b}$, and $\bar{c}$ of the ellipsoidal surface as $\bar{a}^2 = \bar{c}^2(1 + \lambda^2)$, $\bar{b}^2 = \bar{c}^2(1 + \mu^2)$ ($\mu \le \lambda < 1$).

The objective of this paper is the autonomization of system of equations (1), the construction of periodic solutions for the autonomized equations using the Lyapunov and small parameter methods, and the investigation of their stability.

2. AUTONOMIZED EQUATIONS

System of equations (1) represents a system with variable coefficients, and with the help of the following space–time transformation:

$$x = v(t)\xi(\tau), \quad y = v(t)\eta(\tau), \quad z = v(t)\zeta(\tau), \quad d\tau = u(t)dt, \tag{3}$$

where $v(t)$ is a factor and $u(t)$ is the kernel of the transformation which should be determined, is reduced to the system with constant coefficients, i.e., to autonomized equations [2]. The autonomization conditions have the form

$$\frac{1}{u^2 v}\frac{d^2 v}{dt^2} = a_0, \quad \frac{\rho_0(t)}{u^2} = b_0, \quad \frac{\rho_1(t)v^2}{u^2} = b_1, \quad \frac{\rho_2(t)v^4}{u^2} = b_2, \tag{4}$$

and

$$uv^2 = b_3, \quad \Omega v^2 = b_4 \left(\frac{\Omega}{u} = \frac{b_4}{b_3} = \Omega_1\right), \tag{5}$$

where a_0 and b_k ($k = 0, 1, 2, \ldots, 4$) are arbitrary constants, such that $b_0 > 0$, $b_1 > 0$, and $b_2 > 0$, since the densities $\rho_0(t)$, $\rho_1(t)$, and $\rho_2(t)$ are positive functions. In this case, for the factor of transformation (3), the function $v(t)$, as the solution to differential equation (4), the following expression was found in [2]:

$$v^2(t) = \alpha t^2 + 2\beta t + \gamma, \quad \alpha = \text{const}, \quad \beta = \text{const}, \quad \gamma = \text{const}. \tag{6}$$

The functions $u(t)$, $\Omega(t)$, $\rho_0(t)$, $\rho_1(t)$, and $\rho_2(t)$, and the connection between the independent variables τ and t are determined by the equalities

$$u(t) = \frac{b_3}{\alpha t^2 + 2\beta t + \gamma}, \quad \Omega(t) = \frac{b_4}{\alpha t^2 + 2\beta t + \gamma}, \tag{7}$$

and

$$\rho_i(t) = \frac{b_i b_3^2}{(\alpha t^2 + 2\beta t + \gamma)^{i+2}}, \quad (i = 0, 1, 2), \quad \tau = b_3\int\frac{dt}{\alpha t^2 + 2\beta t + \gamma}. \tag{8}$$

Then the differential law of variation of the densities of layers of the elliptical galaxy takes the form

$$\rho_i(t) = -k_i[\rho_i(t)]^{\frac{2i+5}{2i+4}}, \quad k_i = (2i+4)(b_i b_3^2)^{-\frac{1}{2i+4}} > 0, \quad (i = 0, 1, 2). \tag{9}$$

By analogy with the Eddington–Jeans law [5, 6] which was formulated based on the theory of evolution of stars of the main sequence, we assume that the following generalized law of the galaxy mass variation is satisfied:

$$\frac{dM(t)}{dt} = M_0 M^n(t), \quad (M_0 = \text{const}, n \ge 0). \tag{10}$$

Mass $M(t)$ (or density) variation law (10) is called the generalized law, or the analogue of the Eddington–Jeans law. Therefore, differential density variation law (9) corresponds to the generalized law, or the analogue of Eddington–Jeans law (10) for $n = 5/4$, $n = 7/6$, and $n = 9/8$, respectively.

The angular rotation velocity of the galaxy E $\Omega(t)$ should satisfy the Poincare inequality $\Omega^2(t) < 2\pi GK\rho(t)$ [1, 7, 8], which represents the condition of existence of a rotating galaxy as a figure of equilibrium. Here, G is the gravitational constant, K is the constant depending on the shape and dimensions of the elliptical galaxy E (given in [1, 3]), and $0 < K < 1$. Taking into account conditions (4) and (7), the Poincare inequality can be rewritten in the form

$$\Omega(t) \le A, \quad A = \text{const} \ge 0. \tag{11}$$

If autonomization conditions (4) are satisfied, the system of equations (1) takes the form

$$\frac{d^2\xi}{d\tau^2} - 2\Omega_1\frac{d\eta}{d\tau} = \frac{\partial\bar{U}}{\partial\xi}, \quad \frac{d^2\eta}{d\tau^2} + 2\Omega_1\frac{d\xi}{d\tau} = \frac{\partial\bar{U}}{\partial\eta}, \quad \frac{d^2\zeta}{d\tau^2} = \frac{\partial\bar{U}}{\partial\zeta}. \tag{12}$$

Here, the force function (generalized potential) $\bar{U}$ in terms of the new variables is equal to

$$\bar{U} = R_0 + \bar{R}, \quad R_0 = \bar{V}_0 + \bar{V}, \tag{13}$$

where

$$\bar{R} = \frac{\bar{U}_0}{4}(R_1\xi^4 + R_2\eta^4 + R_3\zeta^4$$

$$+2R_4\xi^2\eta^2+2R_5\xi^2\zeta^2+2R_6\eta^2\zeta^2)$$

$$+\frac{\bar{U}_1}{6}(S_1\xi^6+S_2\eta^6+S_3\zeta^6+3S_4\xi^2\eta^4$$

$$+3S_5\xi^4\eta^2+3S_6\xi^2\zeta^4+3S_7\xi^4\zeta^2$$

$$+6S_8\xi^2\zeta^2+3S_9\eta^4\zeta^2+3S_{10}\eta^2\zeta^4), \quad (14)$$

$$\bar{U}_0 = \varepsilon b_1 \ge 0, \quad \bar{U}_1 = \varepsilon^2 b_2 \ge 0,$$

$$\bar{V} = -\frac{1}{2}(a_1\xi^2+a_2\eta^2+a_3\zeta^2),$$

$$\bar{V}_0 = 2\pi G\bar{c}^2\sqrt{(1+\lambda^2)(1+\mu^2)}$$

$$\times\left(b_0-\frac{\varepsilon}{2}b_1-\frac{\varepsilon^2}{3}b_2\right).$$

Note that system of equations (12) for $a_1 = V_{01} - \Omega_1^2$, $a_2 = V_{02} - \Omega_1^2$, and $a_3 = 0$ represents the equations of motion of the star inside the elliptical galaxy with a constant mass in the coordinate system rotating with the constant angular velocity [1], and for $a_1 = a_2 = -\Omega_1^2$, and $a_3 = 0$, inside a spherical cluster. The constants a_k ($k = 1, 2, 3$) included in the expression for $\bar{V}$ are connected with a_0, b_0, and Ω_1 as

$$a_1 = a_0 + b_0V_{01} - \Omega_1^2, \quad a_2 = a_0 + b_0V_{02} - \Omega_1^2,$$
$$a_3 = a_0 + b_0V_{03}, \quad (a_3 \ge a_2 \ge a_1). \quad (15)$$

Therefore, due to condition (4), among nine arbitrary constants a_i ($i = 0, 1, 2, 3$) and b_k ($k = 0, 1, 2, 3, 4$) only six constants are independent: a_0 and b_k ($k = 0, 1, 2, 3, 4$). In relation (15) the inequality in brackets follows from the conditions $b_0 \ge 0$ and $V_{03} \ge V_{02} \ge V_{01}$ [4].

Since $\partial\bar{U}/\partial t = 0$, system of equations (12) admits the analogue of the Jacobi integral [1],

$$\left(\frac{d\xi}{d\tau}\right)^2+\left(\frac{d\eta}{d\tau}\right)^2+\left(\frac{d\zeta}{d\tau}\right)^2 = \bar{U}+h, \quad h = \text{const}, \quad (16)$$

where $\bar{U}$ is determined by equalities (13) and (14). The zero velocity surfaces $\bar{U} = C$ and the domain of possible star motion $\bar{U} \ge C$ ($C = -h$), respectively, can be easily obtained from this integral [1]. We point out that in the third equation of system (13) the additional term with the coefficient a_3 appears.

3. EQUATIONS OF UNPERTURBED MOTION

Let us consider autonomized system of equations (12) for $\varepsilon = 0$, i.e., the equations of unperturbed star motion inside a homogeneous elliptical galaxy with a constant mass. For these equations the analogue of the Jacobi integral has the same form as (16), where $\bar{U}$ should be replaced by R_0.

Obviously, for $a_k \ge 0$ and $\bar{V}_0 + h_0 \ge 0$, or $a_k \le 0$ and $\bar{V}_0 + h_0 \le 0$ ($k = 1, 2, 3$) the unperturbed star motion can take place inside an ellipsoid with the half-axes $\bar{a}_k = \sqrt{(2\bar{V}_0+2h_0)/a_k}$ ($k = 1, 2, 3$). For different signs of a_k and $\bar{V}_0 + h_0 \ge 0$ (or $\bar{V}_0 + h_0 < 0$), the unperturbed star motion takes place inside a hyperboloid of one sheet (two sheets).

The characteristic equation of the system of differential equations of unperturbed motion

$$(\Lambda^4+p\Lambda^2+q)(\Lambda^2+a_3) = 0,$$
$$p = 4\Omega_1^2+a_1+a_2, \quad q = a_1a_2, \quad (17)$$

for $a_k > 0$ ($k = 1, 2, 3$) has pure imaginary roots, $\Lambda_{1,2,3,4} = \pm i\theta_k$ ($k = 1, 2$), and $\Lambda_{5,6} = \pm i\theta_3$, where

$$\theta_1 = \sqrt{\frac{p}{2}-\sqrt{D}}, \quad \theta_2 = \sqrt{\frac{p}{2}+\sqrt{D}},$$
$$\theta_3 = \sqrt{a_3}, \quad D = \frac{p^2}{4}-q. \quad (18)$$

Moreover, due to (15), the condition of imaginary character of roots of Eq. (17) can be rewritten in the form $p \ge 0$, $q \ge 0$, $D \ge 0$. Then if these inequalities are satisfied, the general solution to the system of equations of unperturbed motion can be represented in the form [4, 9]

$$\xi = C_1\cos\omega_1+P_2C_2\sin\omega_2,$$
$$\eta = P_1C_1\sin\omega_1+C_2\cos\omega_2, \quad \zeta = C_3\cos\omega_3, \quad (19)$$

where it is assumed that

$$P_1 = \frac{2\Omega_1\theta_1}{a_2-\theta_1^2} = \frac{a_1-\theta_1^2}{2\Omega_1\theta_1},$$
$$P_2 = \frac{2\Omega_1\theta_2}{\theta_2^2-a_1} = \frac{\theta_2^2-a_2}{2\Omega_1\theta_2}, \quad \omega_k = \theta_k\tau+\sigma_k. \quad (20)$$

Here, C_k and σ_k are arbitrary constants, and θ_k ($k = 1, 2, 3$) are determined by equalities (18). It was shown in [3] that for $\zeta = 0$ orbit (19) represents an elliptical epicycloid.

4. PERIODIC SOLUTIONS CONSTRUCTED USING LYAPUNOV AND POINCARE SMALL PARAMETER METHODS

Let us use the Lyapunov method [3] for construction of periodic solutions to system of equations (12) with the period T in the neighborhood of the central libration point L_1. Let $\Lambda = \pm i\theta$ be the pair of imaginary roots of characteristic equation (17). Then the period T is sought

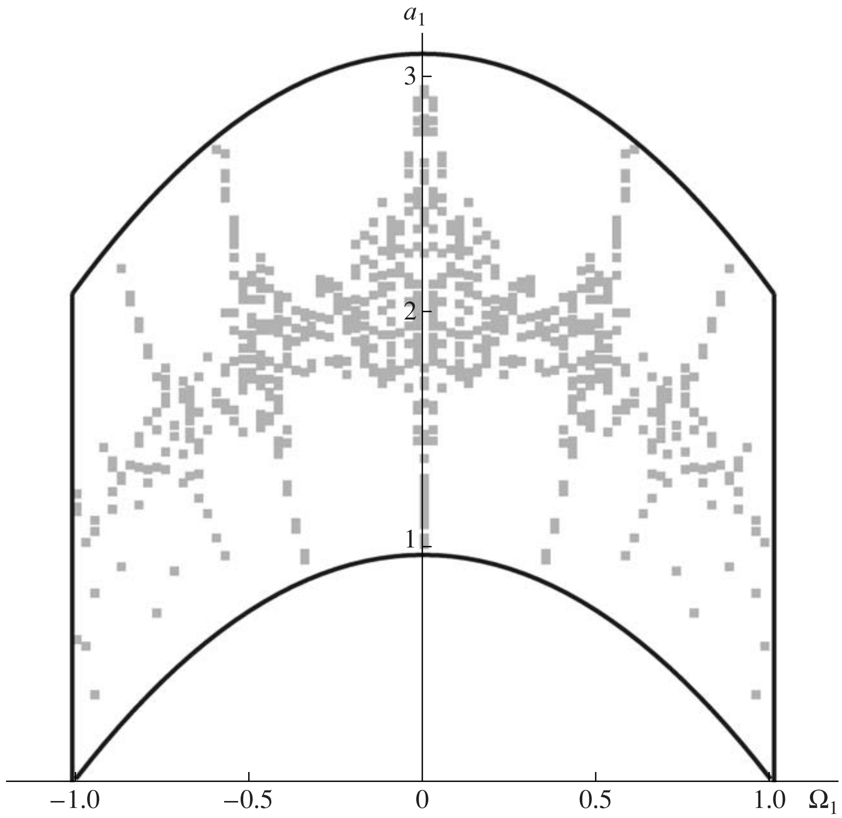

Fig. 1. Domain of stability of periodic solutions with the period $\tilde{T}_1$ for the elliptical galaxy NGC 680 depending on the parameters Ω_1 and a_1.

in the form of the series in powers of the arbitrary constant c,

$$T = \frac{2\pi}{\theta}(1 + h_2c^2 + h_3c^3 + h_4c^4 + h_5c^5 + \ldots), \quad (21)$$

and the new independent variable ν is introduced instead of τ as $\nu = 2\pi(\tau - \tau_0)/T$. Let us try to satisfy the system of equations obtained after this substitution by segments of the series in powers of the arbitrary constant c to 5th power inclusive [10, 11],

$$\xi = \sum_{k=1}^{5}\xi_k c^k + \ldots,$$
$$\eta = \sum_{k=1}^{5}\eta_k c^k + \ldots, \quad \zeta = \sum_{k=1}^{5}\zeta_k c^k + \ldots. \quad (22)$$

These series absolutely converge for sufficiently small c; the procedure of construction of periodic solutions with the help of the Lyapunov method using series (22) was described in detail in [3]. For $|c| < 0.9613$ the families of periodic solutions represent ellipses and straight segments and are situated inside this elliptical galaxy [3]. For fast convergence of series (22) ordered in powers of the arbitrary constant c, a narrower interval of c values can be taken [3].

Now, let us construct periodic solutions to system of equations (12) with the period $\tilde{T}_k = 2\pi/\theta_k$ ($k = 1, 2, 3$) using the Poincare small parameter method. We take the dimensionless parameter ε characterizing the density distribution as the small parameter, and denote by ξ_0 η_0, and ζ_0 the solution to the generating equations for system (12). Then the periodic solutions with the period $\tilde{T}_k$ to this system with are transformed into generating ones at $\varepsilon = 0$, according to the Poincare small parameter method, are sought in the form of absolutely converging for $|\varepsilon| < \varepsilon^*$ ($0 < \varepsilon^* < 1$) series [4, 11, 12],

$$\xi = \xi_0 + \xi_1\varepsilon + \xi_2\varepsilon^2 + \ldots,$$
$$\eta = \eta_0 + \eta_1\varepsilon + \eta_2\varepsilon^2 + \ldots, \quad (23)$$
$$\zeta = \zeta_0 + \zeta_1\varepsilon + \zeta_2\varepsilon^2 + \ldots,$$

where ξ_k, η_k, ζ_k ($k = 1, 2, \ldots$) are the sought periodic functions of τ with the period $\tilde{T}_k$. For construction of these functions we substitute series (23) into system (12) and equate the coefficients at the same powers of the small parameter ε in both sides of these equations.

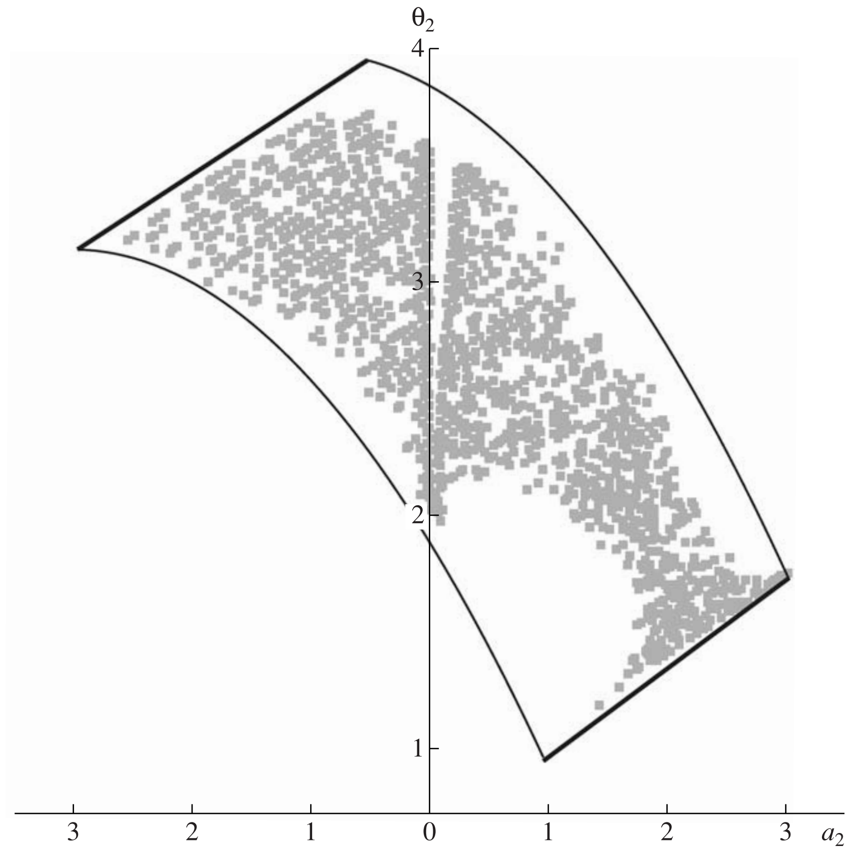

Fig. 2. Domain of stability of periodic solutions with the period $\tilde{T}_2$ for the elliptical galaxy NGC 680 depending on the parameters a_2 and θ_2.

This yields the system of differential equations with respect to ξ_k, η_k, ζ_k. Then we make the following substitution in system (12): $a_k = \alpha_k + \varepsilon\beta_k + \varepsilon^2\gamma_k$, where α_k, β_k, and γ_k ($k = 1, 2, 3$) are some constant nonzero quantities. This substitution guarantees that the periods of solutions (23) do not coincide with the periods of self-oscillations. Thus, resonance relations are eliminated [4, 12]. The procedure of construction of these solutions was described in detail in [4].

5. STABILITY AND COMPARISON OF PERIODIC SOLUTIONS

For investigation of stability in the Lyapunov sense of periodic solutions with the period $\tilde{T}_k$ ($k = 1, 2, 3$) constructed using the Poincare small parameter method we use the results of [4], where this procedure was described in detail. Note that the system of variational equations for these solutions is the system of equations with periodic coefficients and the investigation of roots of its characteristic equation allows one to determine the stability in the Lyapunov sense of the studied periodic solutions. For construction of the characteristic equation it is necessary to determine the initial conditions according to which the system of variational equations will be solved numerically. Let us determine four groups of initial conditions [4] which due to the Floquet theory [13, 14] were used in papers by Danby [15], Bennett [16], and Luk'yanov [17] for investigation of stability of triangular libration points in a bounded elliptical three-body problem, in paper by Luk'yanov and Kochetkova [18] for investigation of stability of Lagrangian libration points in a bounded elliptical photogravitational three-body problem, and in paper by Kochetkova [19] for investigation of stability of straight and triangular libration points in the nonlinear approximation in the spatially bounded elliptical photogravitational three-body problem.

Since system of equations (12) is Hamiltonian (there exists the force function), for stability of the studied family of periodic solutions it is necessary that the calculated absolute values of all roots of the corresponding characteristic equation are equal to unity [11, 12].

In [4], the elliptical galaxy NGC 680 from catalogue [20] was taken as the example, and domains of stability of periodic solutions with the periods $\tilde{T}_1$, $\tilde{T}_2$, and $\tilde{T}_3$ depending on values of key parameters were shown in figures. For brevity, in this paper similar results are presented in the form of three figures.

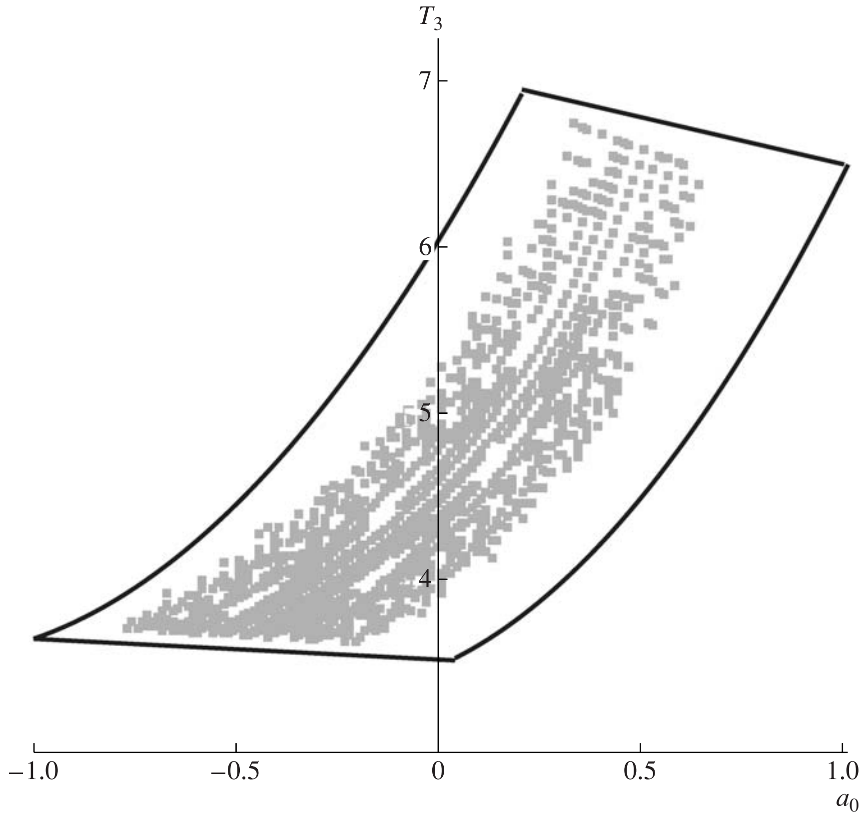

Fig. 3. Domain of stability of periodic solutions with the period $\tilde{T}_3$ for the elliptical galaxy NGC 680 depending on the parameter a_0 and the period $\tilde{T}_3$.

Figure 1 shows the domain of stability of periodic solutions with the period $\tilde{T}_1$ for the elliptical galaxy NGC 680 depending on the parameters Ω_1 and a_1. Figure 2 shows the domain of stability of periodic solutions with the period $\tilde{T}_2$ for this galaxy depending on the parameters a_2 and θ_2. Finally, Fig. 3 shows the domain of stability of periodic solutions with the period $\tilde{T}_3$ for the galaxy NGC 680 depending on the parameter a_0 and the period $\tilde{T}_3$. In Figs. 1, 2, and 3 the domains of stability are not shaded, and the following parameter values are taken into account: $\varepsilon = 0.01$, $b_1 = 0.1$, $b_2 = 0.01$, $0 \le b_0 \le 1$, $|a_0| \le 1$, $|\Omega_1| \le 1$. In this case, the measurement units were chosen in such a way that the gravitational constant and the large half-axis of the elliptical galaxy NGC 680 were equal to unity, $G = 1$, $a = 1$.

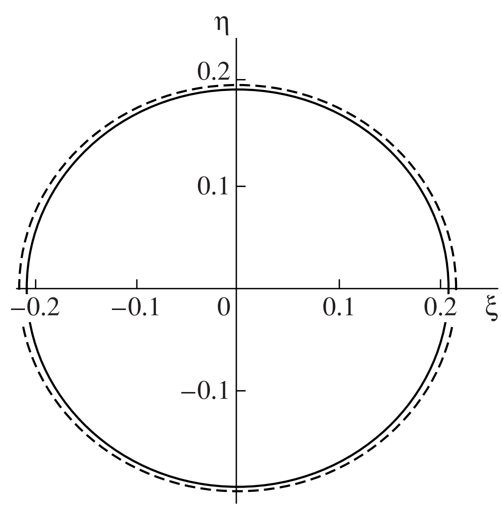

Fig. 4. Periodic solutions constructed using (dashed ellipse) the Lyapunov method and (solid ellipse) the Poincare method for the elliptical galaxy NGC 680.

Figure 4 shows the plots of periodic solutions with the periods T_1 and $\tilde{T}_1$ for the elliptical galaxy NGC 680; the periodic solutions were constructed using the Lyapunov method (dashed ellipse) and the Poincare method (solid ellipse). In this case, for the first periodic solutions the independent variable changes in the interval $[0, T_1]$, where T_1 is determined by equality (21) for $\theta = \theta_1$, and for the second periodic solutions, in the interval $[0, \tilde{T}_1]$, where $\tilde{T}_1 = 2\pi/\theta_1$. Moreover, the values of constants ε, b_1, and b_2 indicated above were taken, and the other constants were taken to be equal to: $a_0 = 0.05$, $b_0 = 0.03$, $\Omega_1 = 0.05$, $c = 0.9613$.

6. CONCLUSIONS

Periodic solutions for the autonomized equations were constructed using the Lyapunov and the Poincare small parameter methods for the problem of a star motion inside a layered inhomogeneous rotating elliptical galaxy with a variable mass. Domains of stability in the Lyapunov sense for periodic solutions constructed using the Poincare method were found. In this case, it was assumed that the density variation laws at the center and in the galaxy layers are different.

ACKNOWLEDGMENTS

We thank Candidate Phys.-Math. Sci., Assistant Professor L. G. Luk'yanov for helpful suggestions and comments.

REFERENCES

1. S. A. Gasanov, Pis'ma Astron. Zh., **33** (12), 925 (2007).
2. S. A. Gasanov, Pis'ma Astron. Zh., **34** (3), 202 (2008).
3. S. A. Gasanov, Pis'ma Astron. Zh., **29** (10), 792 (2003).
4. S. A. Gasanov, Pis'ma Astron. Zh., **32** (3), 217 (2006).
5. J. H. Jeans, Mon. Not. Roy. Astron. Soc., **85**, 912 (1924).
6. E. N. Polyachova, Uchen. Zap. LGU, **424**, 104 (1989).
7. H. Poincare, *Lecons sur les Hypotheses Cosmogoniques* (Paris, 1911).
8. K. F. Ogorodnikov, *Dynamics of stellar systems* (Pergamon, Oxford, 1965; Moscow, Fiz.-Mat. Lit., 1958).
9. N. M. Matveev, *Methods of Integration of Ordinary Differential Equations* (Moscow, Vysshaya Shkola, 1967) [in Russian].
10. A. M. Lyapunov, *Collected Works* (Moscow–Leningrad, 1956) [in Russian].
11. G. N. Duboshin, *Celestial Mechanics. Analytical and Qualitative Methods* (Moscow, 1964) [in Russian].
12. I. G. Malkin, *The Methods of Lyapunov and Poincare in the Theory of Nonlinear Oscillations* (Leningrad-Moscow, Gos. Teh. Teor. Lit., 1949) [in Russian].
13. G. Floquet, S. R. Acad. Sci., **91**, 880 (1880).
14. G. Floquet, Ann. Sci. Ecole Norm. Sup., **12**, 47 (1883).
15. J. M. A. Danby, Astron. J., **69** (2), 165 (1964).
16. A. Bennett, Icarus, **4** (2), 177 (1965).
17. L. G. Luk'yanov, Bull. ITA AN SSSR, **11** (10) (1969).
18. L. G. Luk'yanov and A. Yu. Kochetkova, Vestn. Mosk. Univ., Ser. Fiz., Astron., **37** (5), 71 (1996).
19. A. Yu. Kochetkova, Vestn. Mosk. Univ., Ser. Fiz., Astron., **3** (5), 69 (1999).
20. G. de Vaucouleurs, A. de Vaucouleurs, H. Corwin, et al., *Third Reference Catalouge of Bright Galaxies*, vols. 2, 3, (N.Y., 1991).