ISSN 0027-1349, Moscow University Physics Bulletin, 2010, Vol. 65, No. 5, pp. 372–377. © Allerton Press, Inc., 2010.
Original Russian Text © Yu.V. Ushakov, A.A. Dubkov, 2010, published in Vestnik Moskovskogo Universiteta. Fizika, 2010, No. 5, pp. 38–42.

The Power Spectral Density of the Conditional Markov Pulse Process

Yu. V. Ushakov and A. A. Dubkov

Nizhni Novgorod State University, Nizhnii Novgorod, 603950 Russia

Received February 16, 2010

Abstract—Using the example of a neural system that generates a conditional Markov sequence of delta pulses, the procedure for the derivation of the expression for the spectral power density of such a signal is shown.

Keywords: non-Markov process, hidden Markov process, conditional Markov process, neuron, spike, power spectral density.

DOI: 10.3103/S0027134910050073

INTRODUCTION

Many physical and biological processes may be described pointwise, i.e., in the form of a sequence of random points on the time axis. Typical examples of this are the pulse signals of nerve cells (neurons) that function in the noisy environment of the nervous system [1], as well as the spasmodic behavior of the laser radiation intensity in some operating modes of practical interest [2]. The vitality of spectral analysis of these processes is not doubted. However, the analytical approach in this area causes serious complications if the point process is not a renewal process. In this work, the non-Markov sequence of action pulses (spikes) of the neuron model according to results of [3] can be described by a hidden Markov chain, i.e., it is a conditional Markov process [4]. An analytical expression is proposed for estimating the power spectral density (PSD) of such processes.

The first section of this paper describes the model under study and results of [3] that are necessary for the further presentation. In the second section, a well-known procedure for deducing the general expression for the PSD of point processes is given, whose further development in the third section leads to derivation of the PSD formula for a process with a hidden Markov chain and to the results of using it in the system in question.

1. MODEL

We touch briefly on the results of [3]. The system under study is a model of the neuron ensemble of the auditory analyzer of mammals. It consists of three excited elements, two of which simulate peripheral *sensor neurons* (we shall name them *sensors*) that are effected by the harmonic signals $A_{1,2}\cos\Omega_{1,2}t$, while the third element simulates an *interneuron* (IN) that receives spikes from sensors and generates analogous spikes transmitted to the other neurons. The noise influence of a great number of "neighbors" ($\sim 10^4$ per a neuron) plays an essential part in the functioning of nerve cells; therefore, additive noise sources are used in the model under study. The main object studied in this work is the IN's spike sequence. Due to the presence of noise in the system and the identical form of spikes, this signal is considered as a sequence of random interspike intervals (ISI). It is a non-Markov chain because sensor spikes with a nonexponential ISI-distribution (ISID) density enter the IN input. The system is schematically presented in Fig. 1. As its base element, nondimensional "Leaky Integrate-and-Fire neuron model was used, which is written in the form of the stochastic Langeven equation $\dot{v} = -\mu v + I_{ext}(t) + \xi(t)$, where $v(t)$ is the neuron's membrane

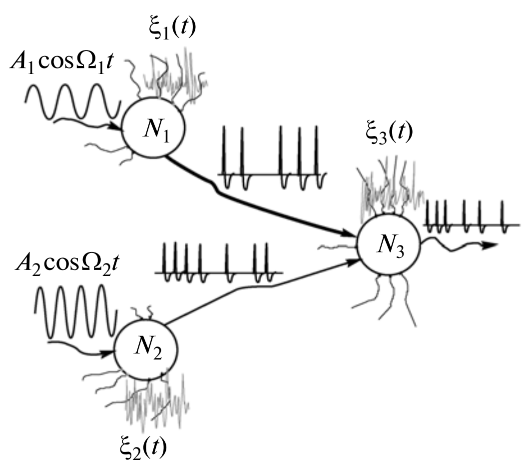

Fig. 1. The investigated model of three neurons.

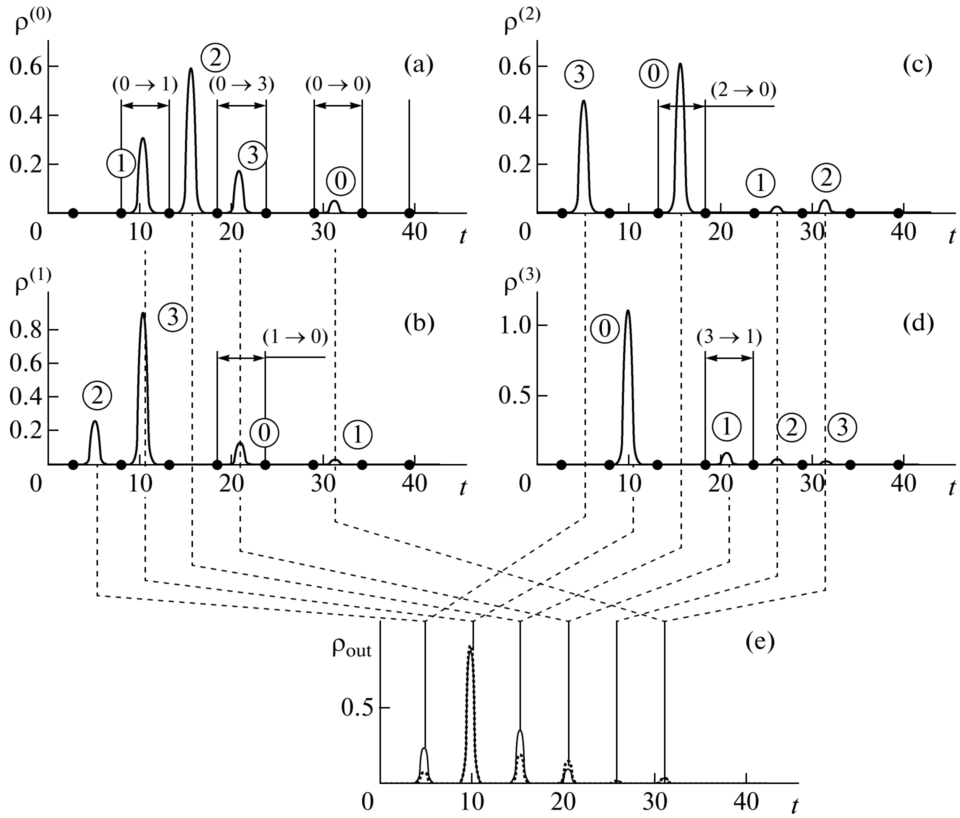

Fig.2. (a–d) The FPTPD of states and (e) the resulting ISID of an interneuron at $\Omega_1 = 0.4$, $\Omega_2 = (3/2)\Omega_1$. The thin dashed line shows the correspondence between the ISID and FPTPD peaks. The digits inside the circles illustrate the state that the IN transfers to if in the current state it generates an ISI that falls within the area of the marked peak. The digits with an arrow inside the brackets $(i \to k)$ also designate some intervals in which the ISI must fall in order to switch the IN from state i to state k. On the bottom panel, the thin solid line corresponds to the theoretical result and the thick dotted line shows the result of numerical simulation.

potential, μ is the relaxation parameter, $I_{ext}(t)$ is the external current, and $\xi(t)$ is the white Gaussian noise. It should be added that for this neuron model a boundary condition exists: when the membrane potential reaches the specified threshold v_{th}, this implies that a spike is generated and $v(t)$ is set to a certain fixed value. Spikes are simulated using Dirac delta functions.

It was shown [3] that when the frequencies of harmonic signals acting on sensors are subject to the ratio of the irreducible fraction $\Omega_2/\Omega_1 = m/n$, the IN may be assumed to be a system with M states ($M = m + n - 1$). At the moment of the spike generation, the IN switches from one state to another and its time resets ("memory" with data on the previous motion is "cleared"). Knowing the ISIDs of sensors, we can to find the first passage time probability distribution (FPTPD) for the generation threshold of the IN membrane potential in every possible state. Any of these FPTPDs is constructed such that every peak and, more precisely, the time interval in whose center the maximum of the peak occurs, corresponds to the transition to another state and unambiguously determines it. As a result, we may illustrate the FPTPD of IN states as shown in Fig. 2, where with the use of the designation $(i \to k)$ the time intervals are marked that are characterized by the fact that when the IN's interspike interval falls within them the IN switches from the state i to the state k. Thus, for the probability of such a switch π_{ik}, we write

$$\pi_{ik} = \int\limits_{(i \to k)} \rho^{(i)}(t)dt$$

and note that it only depends on the FPTPD $\rho^{(i)}(t)$ in the current state, i.e., the given, generally speaking, non-Markov process can be described with the use of a *hidden non-Markov chain* $\{\pi_{ik}\}$. The observed ISI sequence is called a "conditional Markov process" in this case [4, 5].

It is quite understandable that the interneuron's probabilistic ISID $\rho^{out}(\tau)$ (we use τ for designation of a ISI value) is obtained by averaging the FPTPD over the states, for which purpose it is necessary to find stationary probabilities p_i, solving the equation system [6]

$$\begin{cases} \sum_{i=0}^{M-1} p_i \pi_{ik} = p_k, \quad k = 1, 2, \ldots, M-1, \\ \sum_{i=0}^{M-1} p_i = 1. \end{cases}$$

Thus, we find $\rho^{\text{out}}(\tau) = \sum_{i=0}^{M-1} p_i \rho^{(i)}(\tau)$; it is convenient to compare it with the interneuron's ISID, obtained numerically, and to find an acceptable approximation (Fig. 2e) that confirms a certain degree of the validity of the previous reasoning.

2. THE GENERAL EXPRESSION FOR AN PSD

Let us consider a simple procedure for deducing the expression for the PSD of a pulse process using the concrete example of the signal under study, which is a sequence of delta functions:

$$s(t) = \sum_{r=1}^{\infty} \delta(t - t_r),$$

where t_r is the process of generation of the rth spike of the IN.

For this purpose, the signal's complex amplitude $C(j\omega) = \int_0^T s(t)e^{\omega t}\,dt$ is introduced, and then the PSD can be found as the limit [7]

$$S(\omega) = \lim_{T \to \infty} \frac{\langle |C(j\omega)|^2 \rangle}{T}. \quad (1)$$

We consider one process realization of fairly large duration containing a great number of spikes N. It is correctly assumed for the process in question that the length of the mean ISI is $\langle \tau \rangle = T/N$ [8]. Then expression (1) is easily reduced to the following:

$$S(\omega) = \frac{1}{\langle \tau \rangle} \left\{ 1 + \lim_{N \to \infty} \frac{2}{N} \mathrm{Re} \left[\sum_{r=2}^{N} \sum_{q=1}^{r-1} \langle e^{j\omega(t_r - t_q)} \rangle \right] \right\}. \quad (2)$$

In the case of independent ISIs (renewal process), which occurs, e.g., for the ISIs of sensors, the average of the complex exponent in expression (2) breaks down into the product of averages for individual intervals and averaging is performed with the use of the sensor's known ISID $\rho(\tau)$. The double summation is consequently performed by the formula for the geometric progression sum and after simple transformations [7] the following formula is derived:

$$S_{\tau - \langle \tau \rangle}(\omega) = \frac{2}{\langle \tau \rangle} \frac{1 - |\theta(\omega)|^2}{|1 - \theta(\omega)|^2}, \quad (3)$$

where $\theta(\omega) = \int \rho(\tau) e^{j\omega\tau}\,d\tau$ is the characteristic function of the sensor's ISI sequence, and $S_{\tau - \langle \tau \rangle}(\omega)$ is the PSD of signal fluctuations. The integral sign without limits here and below means integration from 0 to ∞.

3. CONDITIONAL MARKOV PROCESS

In the first section, the possibility is shown for regarding the process of the interneuron's ISI generation as a conditional Markov process. In each of the hidden states, the time of the first passage of the generation threshold is known. In this case, generation of some interval τ means switching to the appropriate state. In the general case, the averaging in expression (2) is performed as follows:

$$\langle e^{j\omega(t_r - t_q)} \rangle = \int \ldots \int e^{j\omega(\tau_r + \tau_{r+1} + \ldots + \tau_{q-1})} \times \rho(\tau_r, \tau_{r+1}, \ldots, \tau_{q-1})\, d\tau_r \ldots d\tau_{q-1}, \quad (4)$$

where $\tau_i = t_{i+1} - t_i$ is the ISI and $\rho(\tau_r, \tau_{r+1}, \ldots, \tau_{q-1})$ is the common density of probability of occurrence in succession of the interspike intervals τ_r, τ_{r+1}, etc.

For illustration, we consider the particular case

$$\langle e^{j\omega(t_4 - t_1)} \rangle = \langle e^{j\omega(\tau_1 + \tau_2 + \tau_3)} \rangle = \int e^{j\omega\tau_1} \rho(\tau_1)\, d\tau_1 \times \int e^{i\varphi\tau_2} \rho(\tau_2/\tau_1)\, d\tau_2 \int e^{j\omega\tau_3} \rho(\tau_3/\tau_1, \tau_2)\, d\tau_3. \quad (5)$$

Because the process is stationary, the averaging should not depend on the place on the time axis where the first ISI that enters in the power of the complex exponent is taken, i.e., the average only depends on the number of ISIs in the average: $\langle e^{j\omega(t_4 - t_1)} \rangle = f_3(\omega)$ or in the general case

$$\langle e^{j\omega(t_r - t_q)} \rangle = f_{r-q}(\omega). \quad (6)$$

Thus, we assume that the random value τ_1 is generated in a certain unknown state i. Since all states themselves have specified probabilities of occurrence p_i in the hidden Markov chain, it is necessary to produce the appropriate averaging. Therefore

$$\rho(\tau_1) = \sum_{i=0}^{M-1} p_i \rho^{(i)}(\tau_1) = \rho^{\text{out}}(\tau_1). \quad (7)$$

If the random value τ_1 occurs when the IN is in the ith state, then the interval length τ_1 determines the state k to which the IN switches according to $\rho^{(i)}(\tau_1)$. After switching to the kth state, the random value of the second ISI τ_2 is determined by the density $\rho^{(k)}(\tau_2)$. Thus, for the first conditional probability density in (5), we have

$$\rho(\tau_2/\tau_1) = \begin{cases} \rho^{(0)}(\tau_2), & \text{at } \tau_1 \in (i \to 0), \\ \rho^{(1)}(\tau_2), & \text{at } \tau_1 \in (i \to 1), \\ \ldots, \\ \rho^{(M-1)}(\tau_2), & \text{at } \tau_1 \in (i \to M-1) \end{cases} \quad (8)$$

(for example, see Fig. 2).

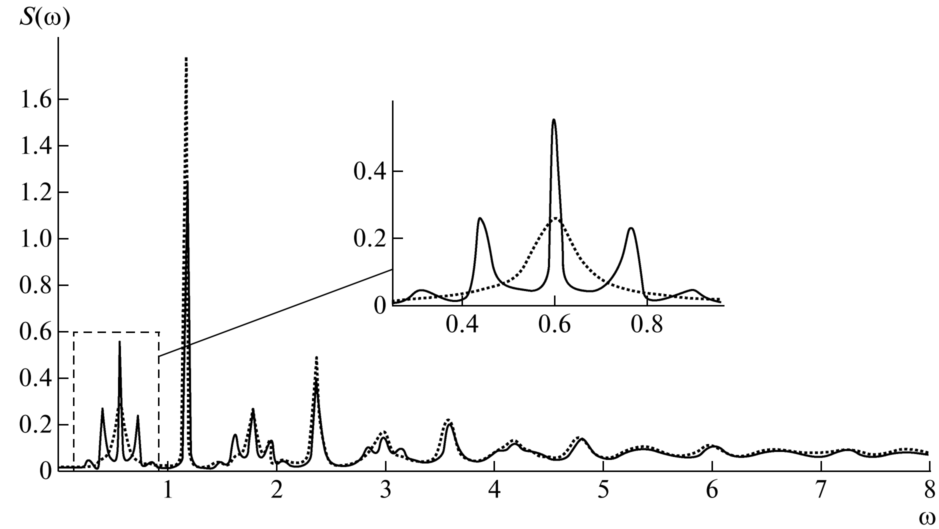

Fig. 3. The PSD of the IN's spike sequence ($\Omega_1 = 0.4$, $\Omega_2 = (3/2)\Omega_1$).

Generally speaking, for a certain $\rho^{(i)}(\tau)$, the ranges of τ values, which switch the IN to a certain state k, are repeated cyclically. However, in the particular cases of interest, the number of nonzero peaks of any $\rho^{(i)}(\tau)$ is less or equal to the number of states M. Therefore, the unambiguous correspondence between peaks and states is supposed. As a result, expression (5) with account for (6)–(8) is rewritten in the following form:

$$f_3(\omega) = \sum_{i,k=0}^{M-1} p_i \int_{(i\to k)} e^{j\omega\tau_1}\rho^{(i)}(\tau_1)d\tau_1 \times \int e^{j\omega\tau_2}\rho^{(k)}(\tau_2)d\tau_2 \int e^{j\omega\tau_3}\rho(\tau_3/\tau_1, \tau_2)d\tau_3. \quad (9)$$

The conditional probability density $\rho(\tau_3/\tau_1, \tau_2)$ is considered in a similar way, from which the independence of variables of integration follows and the ability to write the following occurs:

$$f_3(\omega) = \sum_{i,k,l=0}^{M-1} p_i \int_{(i\to k)} e^{j\omega t}\rho^{(i)}(t)dt \times \int_{(k\to l)} e^{j\omega t}\rho^{(k)}(t)dt \int e^{j\omega t}\rho^{(l)}(t)dt. \quad (10)$$

It is obvious that averaging a complex component with greater number of ISIs in the power is performed simply by addition of the necessary number of operations of summation and appropriate integrals.

By analogy with the definition of a characteristic function, we introduce designations

$$\theta_i(\omega) = \int \rho^{(i)}(t)e^{j\omega t}, \quad \theta_{ik}(\omega) = \int_{(i\to k)} \rho^{(i)}(t)e^{j\omega t}dt.$$

It is obvious that $\theta_i(\omega) = \sum_{k=0}^{M-1}\theta_{ik}(\omega)$. Eventually, expression (10) for the arbitrary number of ISIs n is briefly written in the vector matrix form as

$$f_n(\omega) = \mathbf{p}\boldsymbol{\Theta}^{n-1}(\omega)\boldsymbol{\theta}(\omega), \quad (11)$$

where $\mathbf{p}$ is the vector row with the elements p_0, p_1, ..., p_{M-1}, $\boldsymbol{\Theta}(\omega)$ is the matrix $M \times M$ with the elements $\theta_{ik}(\omega)$, which is raised to the $(n-1)$th power, and $\boldsymbol{\theta}(\omega)$ is the vector-column with the elements $\theta_0(\omega)$, $\theta_1(\omega)$, ..., $\theta_{M-1}(\omega)$. Expression (2) with account for (6) is reduced to the following:

$$S(\omega) = \frac{1}{\langle\tau\rangle}\left\{1 + 2\lim_{N\to\infty}\mathrm{Re}\left[\sum_{n=1}^{N-1}\left(1-\frac{n}{N}\right)f_n(\omega)\right]\right\}. \quad (12)$$

With $\omega \neq 0$, at least one of the norms of the matrix $\boldsymbol{\Theta}(\omega)$, namely $\|\boldsymbol{\Theta}(\omega)\| = \max_i\sum_j|\theta_{ij}(\omega)|$, evidently, does not exceed unity. This condition is sufficient [9] for the convergence of the matrix series in expression (12) at $N \to \infty$. Applying the formula for the summation of the decreasing geometric progression, we derive the final expression for the PSD of the IN's spike consequence

$$S(\omega) = \frac{1}{\langle\tau\rangle}\{1 + 2\mathbf{p}\mathrm{Re}[\mathbf{E} - \boldsymbol{\Theta}(\omega)]^{-1}\boldsymbol{\theta}(\omega)\}, \quad (13)$$

where $\mathbf{E}$ is the unit matrix and $[\mathbf{E} - \boldsymbol{\Theta}(\omega)]^{-1}$ is the inverse matrix for the matrix $[\mathbf{E} - \boldsymbol{\Theta}(\omega)]$.

The solid lines in Figs. 3–5 illustrate the PSD of the IN's spike sequence for different frequency ratios Ω_1/Ω_2. Here, dashed lines show results obtained from formula (3), i.e., on the assumption that the IN's ISID $\rho^{out}(\tau)$ takes place for independent similarly-distributed ISIs and $\theta(\omega) = \int\rho^{out}(t)e^{j\omega t}dt$. It is clearly seen that the curve thus obtained displays the averaged

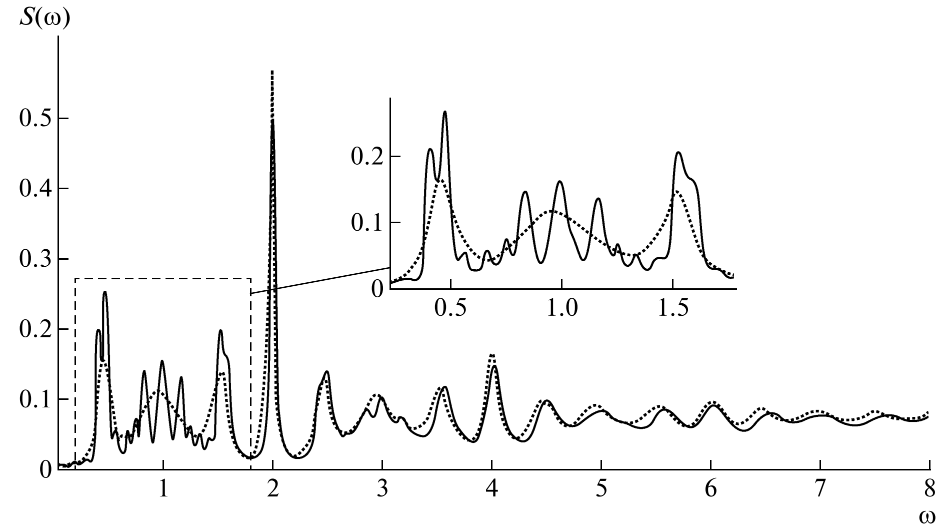

Fig. 4. The PSD of the IN's spike sequence ($\Omega_1 = 0.4$, $\Omega_2 = (5/4)\Omega_1$).

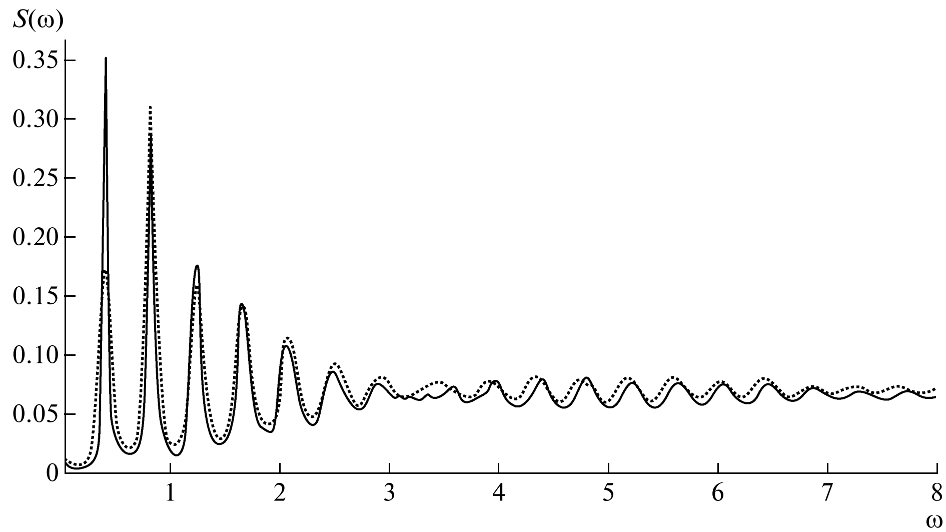

Fig. 5. The PSD of the IN's spike sequence ($\Omega_1 = 0.4$, $\Omega_2 = (16/15)\Omega_1$).

dependence $S(\omega)$ and in some cases (see Figs. 3 and 4) "omits" many details of the PSD in the practically important low-frequency region. However, for the case illustrated in Fig. 5, the solid and dotted lines are very similar; for these plots the frequencies Ω_1 and Ω_2 are weakly distinguished and the PSD peaks are "placed" on frequencies that are multiples of the mean frequency of input harmonic signals $(\Omega_1 + \Omega_2)/2$.

CONCLUSIONS

This work presents a procedure for deducing an expression to estimate the PSD of a conditional Markov process for a very concrete mathematical model. However, this model may be used in a rather wide range of applications because it is easily supplemented with a large number of input and intermediate elements. Similar models find application in the study of sensor neuron systems [10] and the hidden Markov model, as a matter of fact, is widely used for the recognition of speech, graphic symbols, and for other problems of the digital processing of signals [11]. Thus, the results of the analysis performed in this work may be very useful, at least in the mentioned research fields.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, grant no. 08-02-01259-a.

REFERENCES

1. W. Gerstner and W. M. Kistler, *Spiking Neuron Models. Single Neurons, Populations, Plasticity* (Cambridge Univ., Cambridge, 2002).
2. F. T. Arecchi and R. Meucci, Eur. Phys. J. B **69**, 93 (2009).
3. Yu. V. Ushakov, A. A. Dubkov, and B. Spagnolo, Phys. Rev. E **81**, 041911 (2010).
4. R. L. Stratonovich, *Conditional Markov Processes and their Application to the Theory of Optimal Control* (Mosc. Gos. Univ., Moscow, 1965; Elsevier, New York, 1968).
5. L. R. Rabiner, Proc. IEEE **77**, 257 (1989).
6. V. I. Tikhonov and M. A. Mironov, *Markov Processes* (Sov. Radio, Moscow, 1977) [in Russian].
7. R. L. Stratonovich, *Selected Questions of Fluctuation Theory in Radio Engineering* (Sov. Radio, Moscow, 1961) [in Russian].
8. V. I. Tikhonov, *Statistical Radio Engineering* (Sov. Radio, Moscow, 1966) [in Russian].
9. V. I. Priklonskii, *Numerical Methods* (Moscow, 1999) [in Russian].
10. A. Lopera, J. M. Buldu, M. C. Torrent, et al., Phys. Rev. E **73**, 021101 (2006).
11. A. V. Korolev and A. M. Silaev, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **48**, 358 (2005).