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An Equation of Oscillations for the Arrangement of Attracting Solids on the Equilibrium Line of a Balance with Account for Nonlinearity of Up to the Seventh Power

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Abstract—The accuracy of calculating the gravitational constant, G, using a method based on the numerical integration of an equation of oscillations and a method based on the nonlinear oscillation theory is analyzed. Taking account of a higher (the seventh) power at an amplitude of 80 mrad reduces the error of calculating the moment of attraction forces by 47 times. This reduces the error of calculating G from 15 to 0.3 ppm.

Keywords: nonlinear oscillations, equation of oscillations, asymptotic methods, gravitational constant, torsion balance, time-of-swing method, destabilizing factors.

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INTRODUCTION

Asymptotic methods of solving nonlinear equations of oscillations occupy one of the key positions in both radio physics and one of its main directions, i.e., oscillation theory. The model of a nonlinear conservative system is often used in the oscillation theory. Its use is valid when systems with a high Q-factor are considered. This work deals with a sensitive torsion system in which the moment of the attraction of the working solid by probe solids is not proportional to the angle of deviation from the equilibrium position. When terms of higher orders are taken into account, the asymptotic methods provide a better calculation accuracy than the numerical integration of equations of oscillations for interacting spherical solids.

In recent years, some authors have tried to substantially reduce the error of determining the gravitational constant [1–5]. Many of these works have been carried out using a torsion balance, which operates in the free-oscillation mode. The lack of measurement reports and a complex shape of the interacting solids make analysis of these works difficult. The error reported by their authors has reached a value of 15 ppm, at which the calculation accuracy begins to play a significant role. Therefore, increasing the accuracy of calculating the gravitational constant and determining the calculation error are important tasks.

It can clearly be seen from the results of [1, 2] that the problem of the intricacy of calculating the gravitational constant with account for a nonlinear dependence of the moment of attraction on the angle of deviation of the balance was considered. Spherical attracting solids were used. The amplitude of oscillations was limited to a value of ~2 mrad, which made it possible to neglect the nonlinearity. The balance had a Q-factor of 1400 and a period of oscillations of ~535 s. The low sensitivity of the balance and the improper shape of its working solid caused a sharp drop in the period of oscillations when the attractive solids were present. The effect of a destabilizing factor in the form of nonequilibrium flows of rarefied gas ruined that experiment almost completely. Uncertainty in the thickness of the metal coating on the faces of the quartz block led to extra problems. The authors of these works estimated the error at 26.33 ppm.

The authors of [3] did not mention the calculation methods they used and the difficulties that would inevitably arise during the calculations. The static and compensation measurement methods were used. The interacting solids were made of copper with 0.7% Te and had a cylindrical shape. The 30 μm thick, 2.5 mm wide, and 160 mm long suspension of the torsion system was made of beryllium bronze (1.8% Be) and provided a Q-factor of ~10^5. Unfortunately, no data that would make it possible to calculate the attraction moments were reported. The authors estimated the error at 27 ppm.

In [4], the problem was somewhat alleviated by using spherical attracting solids, although it remains fairly challenging because the working solid was a rectangular gold-plated pyrex wafer.

The authors of [4] aimed to substantially reduce the error of determining the gravitational constant, G. The use of a rotation method made the calculations even more difficult. The authors estimated the error at 13.7 ppm.

In [5], measurements were carried out using a compensation method. The working solid was a horizontal copper cylinder with a mass of 500 g suspended on a 1-m long tungsten filament with a cross section of 17×300 μm. Attracting solids with a mass of 27 kg made of stainless steel or copper were shaped as vertical cylinders. They were placed on a rotating table (whose center coincided with the axis of the filament) on both sides of it. The rotating table could be stopped at four positions, at which maximum gravitational forces were achieved. The value of G was calculated from the stress that should be applied to equilibrate the gravitational moment. The calculations of attraction moments were difficult because of the cylindrical shape of the interacting solids; the values of these moments were not reported. The change in the temperature during the entire experiment did not exceed 0.2 K. The authors estimated the error at 40 ppm.

The authors of [6] prepared to carry out difficult calculations using repeated discrete measurements of the angle of deviation of the balance, φ, which made it possible to calculate both the first and the second time derivatives of φ. Nevertheless, serious problems arose when processing the results. The original method used by the authors could be successful, but limited computing functions made it impossible to carry out a high-quality analysis of the data. Instead of the available four positions, the authors had to use only two of them, viz., the nearest and the farthest ones. Moments of attraction were calculated at a only constant value of φ. It was assumed that at an amplitude of 18 mrad the nonlinearity of the moments of attraction could be neglected. At that time, there was no method to perform the calculations at a large number of values of the angle of deflection. Even today, the tabulation of the moments of attraction in the work under discussion presents a serious problem, despite a gross leap in computer speed. No operations on calculating the gravitational constant can be performed without such preliminary work, even if our methods would be used.

When preparing our measurements, we took the intricacy of the calculations to be carried out into account beforehand. Deviations from the spherical shape were strongly prohibited. This allowed us to start developing an analytical method for the calculation of the gravitational constant. The nonlinear oscillation theory was refined. It was initially improved by taking the fifth power of the angle of deviation of the balance into account, which was appropriate. Subsequently, we managed to take the seventh power of the angle of deviation of the balance into account, but did not use this improvement in the calculation program assuming that the fifth power was sufficient. It should be born in mind that the computation means that were previously available were capable of performing calculations only using simple formulas. Remarkable progress in the evolution of computers has left its imprint upon calculation methods. An additional method emerged that made it possible to carry out calculations directly using two differential equations of oscillations. Despite the fact that it was developed much later, it deserves to be mentioned first. This is the only method that can be applied to an analysis of the works in which the shape of the interacting solids differs from spherical. This method is applicable at high angles of deviation of the balance; however, when the shape of the interacting solids differs from spherical, it requires that the moments of attraction be pretabulated. Nonetheless, if these tabulated values are available, the computation time is greater by approximately two orders of magnitude than that required when using the other method, which was called second. The agreement of the results that were obtained when using these two of our methods provided the absence of errors during their development and provided a fairly high calculation accuracy. However, the exact determination of the error of these methods was a difficult task. The aim of this work was to completely employ all the potential of the theory in [7] by expanding the moments of attraction into a series in powers of the angle of deviation φ up to the seventh power, which provided an increase in the calculation accuracy. The error of method 2 in the previous and current versions also had to be estimated.

In method 1, the period of anharmonic oscillations at two positions of the spherical attracting solids is determined using the Runge–Kutta method at two close values of the gravitational constant, G. The value of G that made it possible to superimpose the difference of the reciprocals of squared calculated and experimental periods of oscillations of the balance, was found by liner interpolation.

Method 2 was based on the anharmonic oscillation theory. The moment of the forces of attraction was expanded into a series in odd powers of the angle of deviation of the balance, φ. The period of oscillations was determined using the nonlinear oscillation theory [7], which also made it possible to calculate the gravitational constant using two equations of oscillations when the attracting solids were located at different positions. The formulas that were obtained in [7, 8] with account for the high-power terms differed. A detailed analysis has shown that the apparent difference is due to the fact that in [7] the amplitude of the first harmonic was used, while in [8] the total amplitude of all harmonics was employed. Upon considering this difference, the formulas that were obtained in those works using the example of the mathematical pendulum strictly coincided. This gave grounds to consider these works valid. The higher the powers were of φ that were used in determining the period of oscillations, the better the accuracy of the method was. The theory was refined by taking into account terms of the seventh power, but method 2 was limited by the use of only terms of the fifth power. It was shown in this work that the employment of all the available potential of the theory makes it possible to substantially reduce the error of calculating the gravitational constant.

1. THE MOMENT OF ATTRACTION WITH ACCOUNT FOR TERMS OF THE SEVENTH POWER

The moment of the forces of attraction that act between two spherical weights with the mass m_1 fixed to the ends of the balance arm and two attracting solids with the mass M is written as follows:

$$K_{1i} = 2GMm_1L_i(b_{1ai} + b_{1bi})\sin\varphi, \quad (1)$$

where $b_{1ai} = L_5/(L_5^2 + L_i^2 - 2L_5L_i\cos\varphi + h^2)^{3/2}$; $b_{1bi} = -L_5/(L_5^2 + L_i^2 + 2L_5L_i\cos\varphi + h^2)^{3/2}$; L_5 and L_i are the distances from the axis of rotation to the center of mass of the spherical weight and the attracting ball (the subscript i means the position of the ball); M is the difference in the masses of the attracting ball and the air it forced out; h is the distance from the center of the attracting ball to the horizontal plane, in which the axis of the arm is located; and φ is the angle of the deviation of the arm from the equilibrium position.

The moment of the force of the attraction of the arm by two balls with the mass M is expressed as follows:

$$K_{2i} = 2GMm_2L_i(b_{2ai} + b_{2bi})\sin\varphi, \quad (2)$$

where $b_{2ai} = (L_i^2 + L_6L_i\cos\varphi + h^2)/\{L_6(L_6^2 + L_i^2 + 2L_6L_i\cos\varphi + h^2)^{1/2}(L_i^2\sin^2\varphi + h^2)\}$; $b_{2bi} = (L_i^2 - L_6L_i\cos\varphi + h^2)/\{-L_6(L_6^2 + L_i^2 - 2L_6L_i\cos\varphi + h^2)^{1/2}(L_i^2\sin^2\varphi + h^2)\}$; m_2 is the mass of the arm; and L_6 is the length of the arm. The moment K_{2i} is obtained by integrating Eq. (1) over the length of the arm. It is assumed that the diameter of the arm is negligibly small. The arm is considered as a set of material points, i.e., as a material segment. We note that the value of K_{2i} determined using formula (2) exceeds its true value, which can yield an underestimated calculated value of G. However, the calculation error is small, since $K_{2i} \ll K_{1i}$ and the diameter of the arm is small compared to the distance between the interacting solids. The structure of the moment K_{2i} substantially differs from that of the moment of the attraction of the arm weights K_{1i}. This moment consists of the terms K_{2ai} and K_{2bi}, which differ in the sign at L_6. The first term contains the multiplier b_{2ai} and the second term includes the multiplier $-b_{2bi}$. The sum of these terms yields the total moment of the attraction of the arm, but each term separately does not correspond to the moment of the attraction of each part of the arm. The main drawback of formula (2) is primarily the presence of the multiplier $\sin\varphi$ in the denominator at $h = 0$, which makes the calculations substantially more difficult and can even lead to their failure if the program is improperly written. When φ tends to zero, the moment K_{2ai} also approaches zero, but this can be clearly shown only after K_{2ai} is presented as a series that contains odd powers of the angle of the deviation of the arm.

The moment K_{1ai} is the sum of two moments, which take the interaction of the attracting solids with both the closer and the farther weights of the arm into account:

$$K_{1i} = K_{1ai} + K_{1bi},$$

where $K_{1ai} = 2GMm_1L_ib_{1ai}\sin\varphi$, and $K_{1bi} = 2GMm_1L_ib_{1bi}\sin\varphi$.

Let us transform the denominator of the term b_{1ai} as follows:

$$(L_5^2 + L_i^2 - 2L_5L_i\cos\varphi + h^2)^{-3/2}$$
$$= [(L_5^2 + L_i^2 - 2L_5L_i + h^2)(1 + b_{3ai}y_i)^{-3/2}],$$

where $b_{3ai} = -2L_5L_i/(L_5^2 + L_i^2 - 2L_5L_i + h^2)$ and $y_i = -\varphi^2/2 + \varphi^4/24 - \varphi^6/720$.

We then calculate the term $(1 + b_{3ai}y_i)^{-3/2}$. Since $b_{3ai}y_i \ll 1$, we employ the Newton binomial using terms up to z^3 inclusive in the following formula:

$$(1+z)^n = 1 + nz + n(n-1)z^2/2 + n(n-1)(n-2)z^3/6.$$

In our case, $z = b_{3ai}y_i$, $n = -1.5$, $n(n-1)/2 = 1.875$, and $n(n-1)(n-2)/6 = -2.1875$. Let us introduce the following designations:

$$b_{4ai} = 2Mm_1L_5L_i/(L_5^2 + L_i^2 - 2L_5L_i + h^2)^{-3/2},$$

$$b_{5ai} = -1.5b_{3ai}, \quad b_{6ai} = 1.875b_{3ai}^2,$$
$$b_{7ai} = -2.1875b_{3ai}^3,$$

$$b_{4bi} = -2Mm_1L_5L_i/(L_5^2 + L_i^2 + 2L_5L_i + h^2)^{-3/2},$$

$$b_{5bi} = -1.5b_{3ai}, \quad b_{3bi} = 2L_5L_i/(L_5^2 + L_i^2 + 2L_5L_i + h^2),$$
$$b_{6bi} = 1.875b_{3bi}^2, \quad \text{and } b_{7bi} = -2.1875b_{3bi}^3.$$

In further calculations we shall take into account the fact that, considering the terms of the seventh power of the angle of deviation of the balance φ, the following relationships are true:

$$\sin\varphi = \varphi - \varphi^3/6 + \varphi^5/120 - \varphi^7/5040,$$
$$y_i^2 = \varphi^4/4 - \varphi^6/24,$$
$$\text{and } y_i\sin\varphi = -\varphi^3/2 + \varphi^5/8 - \varphi^7/80.$$

Let us carry out the following transformations:

$$(1 + b_{5ai}y_i)\sin\varphi = \varphi - \varphi^3(1/6 + b_{5ai}/2)$$
$$+ \varphi^5(1/120 + b_{5ai}/8) - \varphi^7(1/5040 + b_{5ai}/80),$$

$$y_i^2\sin\varphi = \varphi^5/4 - \varphi^7/12,$$
$$y_i^3 = -\varphi^6/8, \quad \text{and } y_i^3\sin\varphi = -\varphi^7/8.$$

Using the above-presented expressions, we write the moment K_{1ai} as the following series:

$$K_{1ai} = Gb_{4ai}[\varphi - \varphi^3/6 + \varphi^5/120 - \varphi^7/5040 + b_{5ai}(-\varphi^3/2 + \varphi^5/8 - \varphi^7/80) + b_{6ai}(\varphi^5/4 - \varphi^7/12) - b_{7ai}\varphi^7/8].$$

After the transformations, we obtain the following expression:

$$K_{1ai} = Gb_{4ai}[\varphi - \varphi^3(1/6 + b_{5ai}/2) + \varphi^5(1/120 + b_{5ai}/8 + b_{6ai}/4) - \varphi^7(1/5040 + b_{5ai}/80 + b_{6ai}/12 - b_{7ai}/8].$$

In the term K_{1bi}, the sign at L_5 changes. When two attracting solids have the same masses, we obtain

$$K_{1i} = (b_{4ai} + b_{4bi})G\varphi[b_{4ai}(b_{5ai}/2 + 1/6) + b_{4bi}(b_{5bi}/2 + 1/6)]G\varphi^3 + [b_{4ai}(1/120 + b_{5ai}/8 + b_{6ai}/4) + b_{4bi}(1/120 + b_{5bi}/8 + b_{6bi}/4)]G\varphi^5 + [b_{4ai}(1/5400 - b_{5ai}/80 - b_{6ai}/12 - b_{7ai}/8) + b_{4bi}(1/5400 - b_{5ai}/80 - b_{6ai}/12 - b_{7ai}/8)]G\varphi^7.$$

In this case, the gravitational constant is expressed by the following relationship:

$$G_{ij} = 5\pi^2 J(T_i^{-2} - T_j^{-2})/(b_{1i} + b_{2i} - b_{1j} - b_{2j}),$$

where

$$b_{1i} = b_{4ai} + b_{4bi} + 3e_{1i}\varphi_{0i}^2/4 + \varphi_{0i}^4(3Ge_{1i}^2\omega_0^{-2}/128 + 5e_{2i}/8) + \varphi_0^6(35e_3/64 + 5e_1e_2\omega_0^{-2}/64 - 57e_1^3\omega_0^{-4}/4096),$$

$$\omega_{0i}^2 = 4\pi^2/T_{0i}^2 + Gb_{4ai}/J,$$

$$e_{1i} = -b_{4ai}(b_{5ai}/2 + 1/6) - b_{4bi}(b_{5ai}/2 + 1/6),$$

$$e_{2i} = b_{4ai}(1/120 + b_{5ai}/8 + b_{6ai}/4) + b_{4bi}(1/120 + b_{5bi}/8 + b_{6bi}/4),$$

$$e_{3i} = -b_{4ai}(1/5400 - b_{5ai}/80 - b_{6a}/12 - b_{7a}/8) - b_{4bi}(1/5400 - b_{5ai}/80 - b_{6a}/12 - b_{7a}/8),$$

and the term b_{2i} accounts for the contribution from all sections of the arm, each part of which is conventionally divided into n equal-length sections with the mass of $m_2/2n$. The sections of the arm are considered as point masses. They are at the distance of $L_6(k–0.5)/n$ from the axis of rotation, where k varies from 1 to n. In method 2, formula (2) is not used, but the arm is considered as a chain of point masses, although it is at the expense of a longer computation time. The length of the sections of the arm is approximately equal to its diameter. When performing calculations using method 1, formula (2) is nevertheless preferred, which makes it possible to reduce the computation time by approximately a factor of 50. In the terms with the subscript j, L_j is used instead of L_i.

2. ANALYSIS OF RESULTS

Let us check the results using data from massive 010216. The filename contains the year, month, and day of the beginning of the measurements. The massive 010216.dat was finalized on January 18, 2002. It contains 8508 report strings and has the following parameters: $T_0 = 1721.990$ s, $M = 14083.566$ g, $m_1 = 0.7192$ g, $m_2 = 2.9673$ g, $L_5 = 11.8016$ cm, $L_6 = 11.1636$ cm, $L_1 = 21.1160$ cm, $L_2 = 23.9131$ cm, and $L_3 = 33.7117$ cm (L_1, L_2, and L_3 are the distances from the centers of mass of the attractive solids to the axis of rotation of the balance at three positions, respectively). The steel attractive solids with 152.4 mm in diameter were fixed at the three positions. The averaged periods of oscillations of the balance were 1606.646, 1660.246, and 1707.673 s, respectively. The differences in the periods of oscillations were 53.600 and 47.427 s; the durations of measurement at each position were 0.893, 0.922, and 0.949 h. In the report of the massive, the strings were selected that are convenient for carrying out analysis in a wide range of φ_{0i} and the numbers of these strings did not change. The strings are selected so that the amplitude of oscillations gradually increases. The direct cycles are basically presented, in which measurements start from position 1 that is closest to the balance. As an exception, in strings 441–446 and 3562–3567, the reverse cycles are also presented. This allows one to see an increase in the amplitude of oscillations in the course of the measurement process in the automatic mode. This effect does not influence the calculation accuracy, since each string of the report contains two amplitudes of oscillations. This is caused by the fact that the movement of the attracting solids leads to the parametric pumping of energy into the torsion system. The higher the Q-factor of the system is, the moment of attraction, and the amplitude of oscillations, the more pronounced this effect is. If the onset of the movement of the attractive solids to the next position is delayed, the effect of changes in the amplitude of oscillations of the balance can be neglected.

In this experiment, a suspension filament 15 μm in diameter was used, which was made of the molybdenum–rhenium MR-50 alloy. This alloy has an increased tensile strength, which made it possible to prolong the period of oscillations of the balance; however, it possesses high hysteresis losses compared to tungsten alloys. After the thermomechanical treatment of the suspension filament in a vacuum under loading, the Q-factor reached a value of 5000, while in the case of tungsten filaments treated under the same conditions it increased to 20000.

The error of calculating the moment of the attraction of the balance solid in position 1 (when the dis-

Table 1. Exact (K) and approximate values of the moments of attraction of the balance solid with account for terms of the fifth (K_5) and second (K_7) powers of φ in position 1 of massive 010216 at $M = 14\,083.566$ g, $m_1 = 9.7192$ g, $L = 21.1160$ cm, and $L_5 = 11.8016$ cm, as well as the errors of determining these moments σ_{1k} and σ_{2k}

φ, mrad	2	3	4	6	7	
	$10^{11}K$, N m	$10^{11}K_5$, N m	$10^{11}K_7$, N m	$\sigma_{1k} = (K_5 - K)/K$	$\sigma_{2k} = (K - K_7)/K$	σ_{1k}/σ_{2k}
20.00	1.155971959330	1.155971963482	1.155971959322	3.59E–09	6.76E–12	531.746
30.00	1.730085698282	1.730085769147	1.730085698063	4.10E–08	1.27E–10	323.346
40.00	2.299584276962	2.299584806660	2.299584274135	2.303E–7	1.23E–09	187.340
50.00	2.862981373499	2.862983891872	2.862981352595	8.80E–07	7.30E–09	120.471
52.22	2.987081996500	2.987085407369	2.987081965629	1.14E–06	1.03E–08	110.487
60.00	3.418834339504	3.418843330833	3.418834232145	2.63E–06	3.14E–08	83.750
70.00	3.965753952895	3.965780292346	3.965753524937	6.64E–06	1.08E–07	61.547
71.98	4.072868049376	4.072900038716	4.072867499810	7.85E–06	1.35E–07	58.208
78.02	4.397030468740	4.397086532648	4.397029337172	1.28E–05	2.57E–07	49.545
80.00	4.502413433750	4.502480180511	4.502412017312	1.48E–05	3.15E–07	47.123
89.82	5.018212819539	5.018362121586	5.018208825379	2.98E–05	7.96E–07	37.380
90.00	5.027556614017	5.027708007283	5.027552547656	3.01E–05	8.09E–07	37.231
100.0	5.540005170583	5.540319765014	5.539994737554	5.68E–05	1.88E–06	30.154
120.0	6.522530644243	6.523642180082	6.522477547927	1.70E–04	8.14E–06	20.934
140.0	7.442330133936	7.445547129473	7.442120901126	4.32E–04	2.81E–05	15.375

Table 2. Exact (K) and approximate values of the moments of attraction of the balance solid with account for terms of the fifth (K_5) and second (K_7) powers of φ in position 2 of massive 010216 at $M = 14\,083.566$ g, $m_1 = 9.7192$ g, $L = 23.9131$ cm, and $L_5 = 11.8016$ cm, as well as the errors of determining these moments σ_{1k} and σ_{2k}

φ, mrad	2	3	4	6	7	
	$10^{12}K$, N m	$10^{12}K_5$, N m	$10^{12}K_7$, N m	$\sigma_{1k} = (K_5 - K)/K$	$\sigma_{2k} = (K - K_7)/K$	σ_{1k}/σ_{2k}
20.00	5.902375180495	5.902375187254	5.902375180488	1.15E–9	1.03E–12	1112.00
30.00	8.839824869502	8.839824984864	8.839824869269	1.31E–8	2.64E–11	494.286
40.00	11.76086492408	11.76086578697	11.76086492098	7.34E–8	2.64E–10	278.027
50.00	14.66015108831	14.66015519458	14.66015106523	2.80E–7	1.57E–9	177.930
52.22	15.30031323510	15.30031879771	15.30031320079	3.64E–7	2.24E–9	162.122
60.00	17.53244521663	17.53245989372	1.753244509752	8.37E–7	6.79E–9	123.225
70.00	20.37263987370	20.37268292777	2.037263939887	2.11E–6	2.33E–8	90.673
71.98	20.93075799428	20.93081029889	20.93075738443	2.50E–6	2.91E–8	85.766
78.02	22.62393205764	22.62402381185	2.262393080113	4.06E–6	5.55E–8	73.023
80.00	23.17578169436	23.17589096746	23.17578012113	4.71E–6	6.79E–8	69.458
89.82	25.88778644352	25.88803128941	2.588778200061	9.46E–6	1.72E–7	55.109
90.00	25.93709329940	25.93734158298	25.93708877604	9.57E–6	1.74E–7	54.889
100.0	28.65199358512	28.65251051511	28.65198195860	1.80E–5	4.06E–7	44.461
120.0	33.92532628169	33.92716077339	3.392526685986	5.41E–5	1.75E–6	30.872
140.0	38.96371101277	38.69604739120	3.896347569185	1.37E–4	6.04E–6	22.677

Table 3. Exact (K) and approximate values of the moments of attraction of the balance solid with account for terms of the fifth (K_5) and second (K_7) powers of φ in position 3 of massive 010216 at M = 14 083.566 g, m_1 = 9.7192 g, L = 33.7117 cm, and L_5 = 11.8016 cm, as well as the errors of determining these moments σ_{1k} and σ_{2k}

φ, mrad	2	3	4	6	7	
	$10^{12}K$, N m	$10^{12}K_5$, N m	$10^{12}K_7$, N m	$\sigma_{1k} = (K_5 - K)/K$	$\sigma_{2k} = (K - K_7)/K$	σ_{1k}/σ_{2k}
20.00	1.318533207206	1.318533207361	1.318533207206	1.17E–10	4.79E–14	2443.78
30.00	1.976248249024	1.976248251662	1.976248249022	1.33E–9	1.22E–12	1092.91
40.00	2.632105263849	2.632105283597	2.632105263817	7.50E–9	1.22E–11	614.993
50.00	3.285491303107	3.285491397188	3.285491302868	2.86E–8	7.28E–11	393.599
52.22	3.430146679462	3.430146806950	3.430146679108	3.72E–8	1.03E–10	360.843
60.00	3.935798862421	3.935799199156	3.935798861189	8.56E–8	3.13E–10	273.327
70.00	4.582427197754	4.582428186979	4.582427192715	2.16E–7	1.10E–9	196.307
71.98	4.709971587710	4.709972789900	4.709971581257	2.55E–7	1.37E–9	186.296
78.02	5.097964189956	5.097966301239	5.097964176738	4.14E–7	2.59E–9	159.727
80.00	5.224783612098	5.224786127460	5.224783595568	4.81E–7	3.16E–9	152.168
89.82	5.850855960631	5.850861608202	5.850855914083	9.65E–7	7.96E–9	121.328
90.00	5.862284707292	5.862290434381	5.862284659902	9.77E–7	8.08E–9	120.850
100.0	6.494357595791	6.494369546986	6.494357473986	1.84E–6	1.88E–8	98.117
120.0	7.739986707149	7.740029342237	7.739986082493	5.51E–6	8.07E–8	68.254
140.0	8.957341127132	8.957465905639	8.957338639909	1.39E–5	2.78E–7	50.168

tance from the axis of rotation of the balance to the center of each attracting ball 152.4 mm in diameter is 21.1160 cm) is presented in Table 1. In the entire range of the angle of deviation of the balance φ, the consideration of the terms of the seventh power leads to a substantial decrease in the error. At an amplitude of 80 mrad, the consideration of the terms of the fifth power yields an error of calculating the moment of the attraction of the balance solid of -1.48×10^{-5}. When the terms of the seventh power are taken into account, the error is only 3.15×10^{-7}. At the second position, the balls move away from the balance, which reduces the moment of attraction and the calculation error (see Table 2). At the third position, the distance to the attracting solids substantially increases, which considerably reduces the moments of attraction (see Table 3). Figure 1 shows the errors of calculating the moments at the three positions in a range of 50–100 mrad. These errors finally determine the error of calculating G_{ij}. Table 4 presents the periods and amplitudes of oscillations at each position, as well as the values of the gravitational constant calculated using method 1 (column 8) and method 2 (columns 9 and 10) with account for the terms of the fifth and seventh powers. In Table 4, the error of method 1 is designated σ_1 and that of method 2 with account for the terms of the fifth power is designated σ_2. The error of method 2 in column 10 can be approximately estimated from the error of calculating the moment of attraction, which is substantially smaller than the error that is achieved when taking only the terms of the fifth power into account. Column 10 can be used to estimate the calculation errors in columns 8 and 9, since this error is comparatively small. At values of φ_0 of up to 100 mrad, $\sigma_2 < \sigma_1$, while, with a further increase in φ_0, $\sigma_1 < \sigma_2$. The error of determining G_{12} exceeds the error of determining G_{13}. Among the three available combinations, the

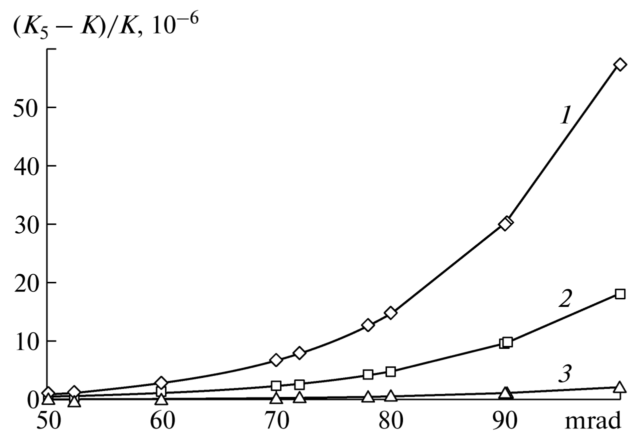

Fig. 1. Error of calculating the moment of the attraction of the balance solid $(K_5 - K)/K$ with account for terms of the fifth power of the angle of deviation of the balance φ at positions *1*, *2*, and *3* (the lower curve).

Table 4. Values of the gravitational constant that were calculated using method 1 (column 8) and method 2 with account for terms at φ_0^4 (column 9) and φ_0^6 (column 10). The calculation error for method 1 is designated as σ_1 and that for method 2 (column 9) is designated as σ_2

1	2	3	4	5	6	7	8	9	10	11	12
N	n_i	n_j	T_i, s	T_j, s	φ_{0i}, mrad	φ_{0j}, mrad	$G_{ij} \times 10^{11}$ (N m^2)/kg^2			σ_1	σ_2
0441	1	2	1607.347	1660.962	46.98	47.78	6.6747326	6.6747468	6.6747506	2.70E–6	5.69E–7
0442	2	3	1660.962	1708.410	47.78	48.19	6.6726179	6.6726255	6.6726265	1.29E–6	1.50E–7
0443	1	3	1607.347	1708.410	46.98	48.19	6.6737890	6.6737995	6.6738022	1.98E–6	4.05E–7
0444	3	2	1708.410	1660.953	48.19	47.95	6.6742030	6.6742102	6.6742113	1.24E–6	1.65E–7
0445	2	1	1660.953	1607.327	47.95	47.11	6.6766578	6.6766706	6.6766746	2.52E–6	5.99E–7
0446	3	1	1708.410	1607.327	48.19	47.11	6.6755616	6.6755725	6.6755751	2.02E–6	3.89E–7
0674	1	2	1607.611	1661.108	52.22	53.12	6.6726337	6.6726533	6.6726606	4.03E–6	1.09E–6
0675	2	3	1661.108	1708.490	53.12	53.58	6.6715742	6.6715829	6.6715848	1.59E–6	2.85E–7
0676	1	3	1607.611	1708.490	52.22	53.58	6.6721607	6.6721756	6.6721805	2.97E–6	7.34E–7
2836	1	2	1608.279	1661.267	71.98	73.27	6.6745053	6.6745611	6.6746120	1.60E–5	7.63E–6
2837	2	3	1661.267	1708.351	73.27	73.94	6.6744077	6.6744203	6.6744336	3.88E–6	1.99E–6
2838	1	3	1608.279	1708.351	71.98	73.94	6.6744630	6.6744997	6.6745337	1.06E–5	5.09E–6
2956	1	2	1608.565	1661.372	78.02	79.41	6.6752060	6.6752867	6.6753706	2.47E–5	1.26E–5
2957	2	3	1661.372	1708.340	79.41	80.14	6.6740354	6.6740537	6.6740753	5.98E–6	3.24E–6
2958	1	3	1608.565	1708.340	78.02	80.14	6.6746839	6.6747357	6.6747917	1.62E–5	8.39E–6
3121	1	2	1609.285	1661.679	89.82	91.50	6.6728554	6.6729201	6.6731144	3.88E–5	2.91E–5
3122	2	3	1661.679	1708.409	91.50	92.39	6.6742756	6.6742820	6.6743329	8.59E–6	7.63E–6
3123	1	3	1609.285	1708.409	89.82	92.39	6.6734939	6.6735344	6.6736643	2.55E–5	1.95E–5
3221	1	2	1609.718	1661.732	99.76	101.60	6.6746631	6.6746968	6.6750647	6.02E–5	5.51E–5
3222	2	3	1661.732	1708.228	101.60	102.57	6.6759000	6.6758918	6.6759877	1.31E–5	1.44E–5
3223	1	3	1609.718	1708.228	99.76	102.57	6.6752222	6.6752377	6.6754832	3.91E–5	3.68E–5
3427	1	2	1611.562	1662.558	123.32	125.56	6.6740600	6.6736591	6.6750021	1.41E–4	2.01E–4
3428	2	3	1662.558	1708.422	125.56	126.74	6.6735300	6.6733883	6.6737345	3.06E–5	5.19E–5
3429	1	3	1611.562	1708.422	123.32	126.74	6.6738264	6.6735418	6.6744344	9.11E–5	1.34E–4

combination G_{23} has the smallest calculation error. We note that the errors of calculating the moments of attraction exceed the errors of calculating the gravitational constant. For example, the errors of calculating the moment of attraction at $\varphi = 71.98$ and 89.82 mrad are as follows: 7.85×10^{-6} and 2.98×10^{-5} in position 1, 2.50×10^{-6} and 9.46×10^{-6} in position 2, and 2.55×10^{-7} and 9.65×10^{-7} in position 3, respectively. At the same amplitudes of oscillations, the errors of determining the other values of G are as follows: 7.63×10^{-6} and 2.91×10^{-5} for G_{12}, 5.09×10^{-6} and 1.95×10^{-5} for G_{13}, and 1.99×10^{-6} and 7.63×10^{-6} for G_{23}. It can be concluded from the data in Tables 1 and 2 that at $\varphi_0 =$ 80 mrad the calculation error for G_{12} will not exceed 0.3 ppm, the calculation error for G_{13} will be ever smaller, and the calculation error for G_{23} will not exceed 0.06 ppm. It follows from the data in Table 3 that if an additional fourth position is available in which no attracting solids exist the error of calculating G_{34} will not exceed 0.003 ppm. The error of calculating G_{12} in a range of φ_0 from 50 to 100 mrad is shown in Fig. 2.

The consideration of nonlinearity is one of the key problems in calculations of the gravitational constant. In position 1 at $\varphi = 20$ mrad the deviation from the linear law of the moment of attraction reaches 1786 ppm, which is almost the same as the deviation in position 1 in [6] when φ reaches 18 mrad. With an increase in φ, nonlinearity rapidly grows. For example, at $\varphi =$ 50 mrad, the deviation from the linear law reaches 11205 ppm, while, at $\varphi = 100$ mrad, it is as high as

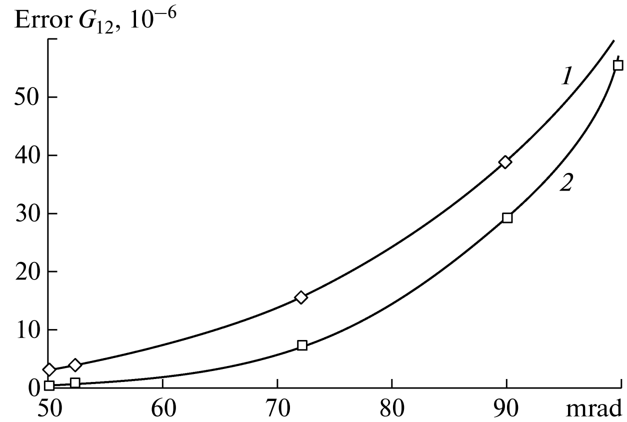

Fig. 2. The error of calculating the gravitational constant G_{12} using methods *1* and *2* (the lower curve) with account for terms of the fourth power of the amplitude of oscillations of the balance.

45096 ppm. In position 2, at the same values of φ, the deviations are 1218, 7638, and 30687 ppm, which approximately correspond to the deviations in position 3 in [6]. In position 3, at the same values of φ, the deviations are 563, 3526, and 14135 ppm.

CONCLUSIONS

Accounting for terms of up to the seventh power of φ leads to a large drop in the errors of calculating both the moments of the attraction forces and the gravitational constant for all combinations of the positions of the attracting balls [9]. The refined version of method 2 makes it possible to determine the true error of both method 2, in which terms at φ_0^6 have not yet been used, and method 1. The error reaches a value of 0.3 ppm, which is 45 times smaller than the error obtained in [4] (13.7 ppm). Therefore, our calculation methods will not introduce an error in the determination of the gravitational constant, even in the works in which smaller errors are reported. The use of a shape of the interacting solids that is not spherical makes it impossible to use the simpler and more reliable method 2, as well as making the use of method 1 more difficult. In this case, method 1 can be implemented only after the moments of attraction are precalculated.

In method 2, the previously achieved computation time (which is shorter by almost two orders of magnitude than that achieved in method 1) was almost maintained. The possibility arose to widen the range of φ_{0i}, in which a small calculation error is provided. However, a substantial growth in nonlinearity with increasing amplitude of oscillations restricts the upper limit at approximately the previously achieved level (~80 mrad). With a further increase in the amplitude, requirements for the accuracy of measuring it become much stricter.

In all the experiments, a higher value of φ_0 (100 mrad) is used in the file of the constants instead of the value $\varphi_0 = 80$ mrad, which assigns the upper limit of calculations using method 2. The special program key performs automatic transition from method 2 to method 1 when φ_0 begins to exceed the specified value. At a reported error of 75 ppm [9], an increase in φ_0 in most of the available measurement massives does not substantially reduce the accuracy of determining *G*. However, in some massives that contain large values of φ_0, the refinement of *G* was observed when the nonlinearity of higher orders was taken into account. When any parameter in the file of the constants is varied, new calculations of the massives that contain almost 100000 report strings are carried out in a few hours when this program key is used.

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